6. Curvature of submanifolds, Lie groups

6.1. Liouville's theorem.

Let X be the geodesic vector field on TM.

1. Prove that $\operatorname{div}(G) = 0$.
2. Prove that the geodesic flow preserves the volume of TM.

6.2. Totally geodesic submanifolds.

Prove the following proposition.
For a submanifold $M \subset \tilde{M}$ the following four statements are equivalent:

1. vanishing second fundamental form, i.e. $\kappa(X, Y) = 0$ for all $X, Y \in TM$
2. every geodesic in M is also a geodesic in $\tilde{M}$
3. if $v \in T_pM$ then the unique geodesic $\tilde{\gamma}$ of $\tilde{M}$ with initial velocity v lies initially in M
4. if $c : I \to M$ is a curve, then every ∇-parallel vector field $Y \in \Gamma(c^*(TM))$ is also $\tilde{\nabla}$ parallel.

6.3. Pull-back connections and curvature.

1. Prove the following Proposition.
   Let R be the curvature tensor of a connection ∇ on TM. For another manifold N and a smooth map $F : N \to M$, let R^F denote the curvature tensor of the connection ∇^F along F, given by
   $$R^F(X, Y)W := \nabla^F_X \nabla^F_Y W - \nabla^F_Y \nabla^F_X W - \nabla^F_{[X,Y]} W \in \Gamma(F^*TM)$$
   for $X, Y \in \Gamma(TN)$ and $W \in \Gamma(F^*TM))$. Then
   $$(R^F(X, Y)W)_p = R(F_*X_p.F_*Y_p)W_p$$
   for all $p \in N$.
   Remark: the notation for the pull-back connection along a curve c used in the lectures is ∇_{∂_s} (∇^c in the notation of the proposition).
2. If $F\colon (-1, 1)_s \times (0, 1)_t \to M$ is a smooth map, then $[D/ds, D/dt]V = R(V, \gamma')V$ for $V(t) = \partial_s F(0, t)$, $\gamma'(t) = \partial_t F(0, t)$.

6.4. Lie groups.

Definition: A Lie group G is a smooth manifold G which has a smooth group structure (m and i are smooth maps between smooth manifolds)

$$m : G \times G \to G \quad m(p, q) = pq$$

$$i : G \to G \quad i(p) = p^{-1} .$$

For any $p \in G$, the left-multiplication $L_p : G \to G$ defined by $L_p(q) = pq$ is a diffeomorphism. A Riemannian metric g on G is called left-invariant if $g(v, w) = g(dL_p(v), dL_p(w))$ for all $v, w \in TG$ and $p \in G$. One defines right-multiplication and right invariance analogously. A metric is called bi-invariant if it is both right- and left-invariant.
The Lie algebra $T_eG =: \mathcal{G}$ can therefore be identified with the set of left-invariant vector fields on G.

1. Let $X \in \mathcal{G}$ be a left-invariant vector field. Prove that the flow ϕ_t of X is defined globally for all times t and that the flow lines $\gamma : \mathbb{R} \to G$ with $\gamma(0) = e$ satisfy $\gamma(t + s) = \gamma_t \cdot \gamma_s(a)$.
   Remark: a map $\gamma : \mathbb{R} \to G$ is called a 1-parameter subgroup of G if it satisfies $\gamma(t + s) = \gamma(t) \cdot \gamma(s)$.

2. Prove that if G has a bi-invariant metric g, then the geodesics starting at e are 1-parameter subgroups of G.

Let G be a Lie group with bi-invariant metric g. Let $X, Y, Z \in \Gamma(TG)$ be left-invariant vector fields.

3. Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.
   Hint: show that $\nabla_X X = 0$ for left invariant vector fields.

4. Prove using (3.) that $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.

5. Prove that if X and Y are orthonormal, the sectional curvature of the plane Π generated by X and Y is given by $K(\Pi) = \frac{1}{4}g([X, Y], [X, Y])$. Conclude that if the flows of X and Y commute, the planes Π have vanishing sectional curvature.
   Remark: this shows that the sectional curvature of a Lie group with bi-invariant metric is non-negative.