7. Curvature, conjugate points

7.1. Codazzi equation.

Let $M \subset \bar{M}$ be a submanifold of the Riemannian manifold $(\bar{M}, \bar{g})$. For the second fundamental form K of M, we define

$$(D_X^{\perp} K)(Y, W) := (\bar{D}_X(K(Y,W))^{\perp} - K(D_X Y, W) - K(Y, D_X W),$$

where $W, X, Y \in \Gamma(TM)$. Show that the Codazzi equation

$$\left(\bar{R}(X,Y)W\right)^{\perp} = (D_X^{\perp} K)(Y,W) - (D_Y^{)}(X,W)$$

holds for all $W, X, Y \in \Gamma(TM)$.

7.2. Sectional curvature of submanifolds.

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with sectional curvature $\bar{K}$. Let $p \in \bar{M}$ and $L \subset T\bar{M}_p$ an m-dimensional linear subspace.

1. Prove that there is some $r > 0$ such that for the open ball $B_r(0) \subset T\bar{M}_p$, the set $M := \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\bar{M}$.
2. Let g be the induced metric on M and let K be the sectional curvature of M. Show that for $E \subset TM_p$, we have $K_p(E) = \bar{K}_p(E)$ and if L is a 2-dimensional subspace, then $K \leq \bar{K}$ on M.

7.3. Small balls and scalar curvature.

Let p be a point in the m-dimensional Riemannian manifold (M, g). The goal is to prove the following Taylor expansion of the volume of the ball $B_r(p)$ as a function of r:

$$\operatorname{vol}(B_r(p)) = \omega_m r^m \left(1 - \frac{1}{6(m+2)} \operatorname{scal}(p) r^2 + O(r^3)\right).$$

1. Let $v \in TM_p$ with $|v| = 1$, define the geodesic $c(t) := \exp_p(tv)$ and let $e_1 = v, e_2, \ldots, e_m \in TM_p$ be an orthonormal basis. Consider the Jacobi fields Y_i along c with $Y_i(0) = 0$ and $\dot{Y}_i(0) = e_i$ for $i = 2, \ldots m$. Show that the volume distortion factor of $\exp_p$ at tv is given by

$$J(v,t) := \sqrt{\det\left(\langle T_{tv} e_i, T_{tv} e_j \rangle\right)} = t^{-(m-1)} \sqrt{\det\left(\langle Y_i, Y_j \rangle\right)},$$

where $T_{tv} \coloneqq (d\exp_p)_{tv}$.

2. Let $E_2, \dots, E_m$ be parallel vector fields along c with $E_i(0) = e_i$. Then the Taylor expansion of Y_i is

$$Y_i(t) = tE_i - \sum_{k=2}^{m} \left(\frac{t^3}{6} R(e_i, v, e_k, v) + O(t^4) \right) E_k.$$

3. Conclude that $J(v,t) = 1 - \frac{t^2}{6}\mathrm{Ric}(v,v) + O(t^4)$.

   Hint: Use $\det(I_m + \epsilon A) = 1 + \epsilon \mathrm{trace}(A) + O(\epsilon^2)$.

4. Prove the above formula for $\mathrm{vol}(B_r(p))$.