# 8. Isometries, translations, geodesics and conjugate points

### 8.1. Nearby conjugate points.

Prove the following Lemma.

Suppose $\gamma\colon [0,1] \to M$ is a geodesic and $t_0 \in (0,1)$ is such that $\gamma(t_0)$ is conjugate to $\gamma(0)$ along $\gamma$. Then there exists $\epsilon > 0$ so that the following holds: if $c\colon [0,1] \to M$ is a geodesic with $d(\gamma(t), c(t)) < \epsilon$ for all $t \in [0,1]$, then there exists $t_1 \in (0,1)$ so that $c(t_1)$ is conjugate to $c(0)$ along $c$.

### 8.2. Locally symmetric spaces.

Let $M$ be a connected $m$-dimensional Riemannian manifold. Then $M$ is called *locally symmetric* if for all $p \in M$ there is a normal neighborhood $B(p,r)$ such that the *local geodesic reflection* $\sigma_p \coloneqq \exp_p \circ(-\operatorname{id}) \circ \exp_p^{-1}\colon B(p,r) \to B(p,r)$ is an isometry.

1. Show that if $M$ is locally symmetric, then $DR \equiv 0$.
   *Hint:* Use that $d(\sigma_p)_p = -\operatorname{id}$ on $TM_p$.

2. Suppose that $DR \equiv 0$. Show that if $c\colon\ [-1,1] \to M$ is a geodesic and $\{E_i\}_{i=1}^m$ is a parallel orthonormal frame along $c$, then $R(E_i, c')c' = \sum_{k=1}^m r_i^k E_k$ for *constants* $r_i^k$.

3. Show that if $DR \equiv 0$, then $M$ is locally symmetric.
   *Hint:* Let $q \in B(p,r), q \neq p$, and $v \in TM_q$. To show that $|d(\sigma_p)_q(v)| = |v|$, consider the geodesic $c\colon\ [-1,1] \to B(p,r)$ with $c(0) = p, c(1) = q$, and a Jacobi field $Y$ along $c$ with $Y(0) = 0$ and $Y(1) = v$. Use 2..

### 8.3. Poincaré models of hyperbolic space.

Let us introduce the following two well-known models of the hyperbolic space:

$$\text{Unit ball } \{|z| < 1\} \subset \mathbb{R}^n \text{ equipped with metric } g_{ij} = \frac{4\delta_{ij}}{(1-|z|^2)^2}$$

and

$$\text{Half space } \{x^n > 0\} \subset \mathbb{R}^n \text{ equipped with metric } g_{ij} = \frac{\delta_{ij}}{(x^n)^2}.$$

1. Show that composing the transformations $y = x + (\frac{1}{2} - 2x^n)\boldsymbol{e}_n$ and $z = \boldsymbol{e}_n + (y - \boldsymbol{e}_n)|y - \boldsymbol{e}_n|^{-2}$ give an isometry between the two previous Riemannian manifolds

2. Show that, for the second model, circular arcs at $\{x^n = 0\}$ are geodesics.

3. Show that given any given point all geodesic rays $x(t)$, $t \geq 0$ emanating from it are minimizing up to arbitrarily large values of $t > 0$ (note that this is stronger than geodesic completeness).

4. Show that the sectional curvatures are constantly equal to $-1$.

### 8.4. Translations.

Suppose that $\Gamma$ is a group of translations of $\mathbb{R}^m$ that acts freely and properly discontinuously on $\mathbb{R}^m$.

1. Show that there exist linearly independent vectors $v_1, \ldots, v_k \in \mathbb{R}^m$ such that

$$\Gamma = \left\{ x \mapsto x + \sum_{i=1}^{k} z_i v_i : (z_1, \ldots, z_k) \in \mathbb{Z}^k \right\} \simeq \mathbb{Z}^k.$$

2. Let $l$ denote the infimum of the lengths of all closed curves in $\mathbb{R}^m/\Gamma$ that are not null-homotopic. Show that $l$ equals the length of the shortest non-zero vector of the form $\sum_{i=1}^{k} z_i v_i$ with $z_i \in \mathbb{Z}$ as above.