9. Space forms, curvature

9.1. Complete submanifolds and smooth graphs.

1. Let (M, g) be a complete Riemannian manifold, and let $N \subset M$ be a closed subset which is a submanifold. Denote by h the induced metric on N. Show that (N, h) is complete.

2. Show that graphs of smooth functions $F\colon \mathbb{R}^n \to \mathbb{R}$ are complete submanifolds of $\mathbb{R}^{n+1}$.

9.2. Conjugate points in manifolds with curvature bounded from above.

1. Prove that there are no conjugate points in manifolds with non-positive sectional curvature.

2. Show that in manifolds with sectional curvature at most κ, where $\kappa > 0$, there are no conjugate points along geodesics of length $< \pi/\sqrt{\kappa}$.

3. Show that if $c\colon\ [0, \pi/\sqrt{\kappa}] \to M$ is a unit speed geodesic in a manifold with $\sec \geq \kappa > 0$, then some $c(t)$ is conjugate to $c(0)$ along $c|_{[0,t]}$.

9.3. Diffeomorphisms of hyperbolic space.

Fix any point $p \in \mathbb{H}^m$. Show that $\exp_p\colon T_p\mathbb{H}^m \to \mathbb{H}^m$ is a diffeomorphism.

9.4. On the completeness assumption in the Killing-Hopf Theorem.

The goal of this exercise is to show that there exist non-complete smooth manifolds of constant sectional curvature that are not isometric to a subset of a space form.[1]

1. Let (M, g) be the universal cover of $\mathbb{S}^2 \setminus \{p_1, p_2\}$ with the metric induced by the (smooth) covering map. Verify that M has constant sectional curvature.

2. You can assume that M is diffeomorphic to $\mathbb{R}^2$.

3. Assume that there exists an isometric embedding $i : M \to S$ for a space form S. Show that S admits a covering

$$\pi : \mathbb{S}^m \to S\,.$$

[1] this exercise relies on some topological results that are not necessarily covered in the course

4. Prove (or skip the proof and assume) that i can be lifted to an injective isometry

$$\widetilde{i} : M \to \mathbb{S}^m \quad \pi \circ \widetilde{i} = i\,.$$

5. Conclude by showing that the map $\widetilde{i}$ cannot exist.