1. Riemannian metrics, isometries, Lie derivatives

1.1. Equivalence of definitions of a surface.

Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface. Show that for all $p \in \Sigma$ there exists an open neighborhood $U \subset \mathbb{R}^3$ of p so that $\Sigma \cap U$ is the graph of a smooth function $F \colon V \subset \mathbb{R}^2 \to \mathbb{R}$, i.e. $\Sigma \cap U = \{(x^1, x^2, F(x^1, x^2))\}$ or $\{(x^1, F(x^1, x^3), x^3)\}$ or $\{(F(x^2, x^3), x^2, x^3)\}$.

1.2. Lie derivative.

Let M be a smooth manifold.

1. Let $D \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ be a derivation, i.e. D is $\mathbb{R}$-linear and satisfies $D(fg) = fD(g) + gD(f)$ for all $f, g \in \mathcal{C}^\infty(M)$. Show that there exists a unique vector field $V \in \Gamma(TM)$ so that $D(f) = Vf$ for all $f \in \mathcal{C}^\infty(M)$.
2. Let $V, W \in \Gamma(TM)$. Show that $[V, W] \in \Gamma(TM)$ where $[V, W] \colon f \mapsto VWf - WVf$. Express V in a local chart (φ, U) as $V = V^i \partial_{\varphi^i}$ where $V^i \in \mathcal{C}^\infty(\varphi(U))$, similarly $W = W^j \partial_{\varphi^j}$. Is $[V, W]$ tensorial in V, i.e. $\mathcal{C}^\infty(M)$-linear in V?

1.3. Existence of Riemannian metrics.

Let M be a smooth manifold. Show that there exists a Riemannian metric g on M.
Hint. Use a partition of unity. Carefully check that your construction yields something positive definite!

1.4. Isometries.

1. Show that the map

$$\Big((0, \infty) \times (0, 2\pi), \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2\Big) \ni (r, \phi) \overset{F}{\mapsto} (r\cos\phi, r\sin\phi) \in \Big(\mathbb{R}^2, (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\Big)$$

   is a local isometry.
2. Show that Möbius transformations $z \overset{A}{\mapsto} \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$, are isometries of $(\mathbb{H}^2, g)$ where we define $\mathbb{H}^2 := \{z \in \mathbb{C} \colon \Im z > 0\}$ and $g_{x+iy} = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. Show that $(\mathbb{H}^2, g)$ is homogeneous and isotropic.

1. Solutions

Solution of 1.1: Fix a point $p \in \Sigma$. Let $x_i : \Sigma \to \mathbb{R}$ denote the restriction of the standard Euclidean ambient coordinates (that is for $p = (p_1, p_2, p_3)$, $x_i(p) = p_i \in \mathbb{R}$).
Claim: given $p \in \Sigma$ there exists a pair (i, j), $i \neq j$, $1 \leq 1, j \leq 3$ of coordinates such that $\varphi(i, j) = (x_i, x_j) : \Sigma \to \mathbb{R}^2$ is a smooth chart.
Proof of claim: Given the tangent space $T_p\Sigma \subset \mathbb{R}^3$ there exist (i, j) as above such that the orthogonal projection : $T_p\Sigma \to \mathrm{span}(e_i, e_j)$ is bijective. (Here e_i, e_j denote the standard coordinate vectors in $\mathbb{R}^3$). The corresponding chart $\varphi(i, j)$ has rank 2 at p and by the Implicit Function Theorem it is a local diffeomorphism, hence a chart. Denote by $U \subset \Sigma$ the open set on which $\varphi(i, j)$ is a diffeomorphism.
Pick the chart $\varphi(i, j)$ given above. Let $k \neq i, j$ be such that e_k is orthogonal to $\mathrm{span}(e_i, e_j)$. Then, denoting by (y_1, y_2) the coordinates of $\mathrm{span}(e_i, e_j)$, and defining the function $F : \varphi(i, j)(U) \to \mathbb{R}$ as $F(y_1, y_2) = x_3(\varphi(i, j)^{-1}(y_1, y_2))$,

$$\left\{(y_1, y_2, F(y_1, y_2))\Big|(y_1, y_2) \in \varphi(i, j)(U)\right\} = \Sigma \cap U.$$

Remarks: dim(Σ)=2 does not play a role in the statement, one can prove in the same way that embedded smooth surfaces are locally a graph.
Bonus: how would the statement need to be modified if the surface is just assumed to be immersed?

Solution of 1.2: Recall that for a vector field $V \in \Gamma(TM)$, a point $p \in M$ and a chart $\varphi : U \to \mathbb{R}^m$ satisfying $\varphi(p) = 0$, we write $V_p = v^i \frac{\partial}{\partial \varphi_i}\Big|_p$ and

$$Vf(p) = \frac{d}{dt}\Big|_{t=0} (f \circ \varphi^{-1})(tv) \in \mathbb{R}.$$

Choosing $f = \varphi_i$,

$$V\varphi_i(p) = v^i\,.$$

This motivates the following definition: given a derivation D, on the chart domain U set

$$v^i_D(q) = D(\varphi^i)(q) \text{ for all } q \in U.$$

and

$$V_D(q) = v^i_D(q) \frac{\partial}{\partial \varphi_i}\Big|_q$$

Then, for $f \in C^\infty(U)$, and assuming for now $U \subset \mathbb{R}^m$ is open and convex (not really needed), for $0, x \in U$, the function f can be expressed as a sum:

$$f(x) = f(0) + x^i(x)g_i, \quad g_i = \frac{\partial f}{\partial x_i}(0) \ i = 1, ..., m\,.$$

Applying the derivation D, using its properties and letting $x \to 0$,

$$Df(0) = (Dx^i)(0)g_i = v_D^i(0)\frac{\partial f}{\partial x_i}(0)\,.$$

Since the statement is translation invariant, we proved so far that

$$D = v_D^i \frac{\partial}{\partial x_i} =: V_D \text{ on } U \subset \mathbb{R}^m\,,$$

that is the derivation D is given by $V = V_D$ on U. To obtain the result on $U \subset M$ and then globally on M, use the fact that the derivation on M can be pulled back to a derivation on $U \subset \mathbb{R}^m$ using a chart and a partition of unity.
Uniqueness: Assume $D = V_1 = V_2$ for two vector fields $V_1, V_2 \in \Gamma(TM)$. Then locally in a coordinate chart, $v_1^i \frac{\partial}{\partial x_i} = v_2^i \frac{\partial}{\partial x_i}$, which determines the vector field uniquely, i.e. $V_1 = V_2$.

Part 2. By part 1, we know that the space of derivations is in 1-to-1 correspondence with the space of smooth vector fields (although strictly speaking, we did not show the easier implication that vector fields are derivations in part 1). To show $[V, W] \in \gamma(TM)$ it suffices to show that it satisfies the derivation property:

$$\begin{aligned}
[V, W](fg) &= VW(fg) - WV(fg) = V(W(f)g + fW(g)) - W(V(f)g + fV(g)) \\
&= V(W(f))g + fV(W(g)) - fW(V(g)) - W(V(f))g \\
&= ([V, W]f)(g) + f([V, W]g)\,,
\end{aligned}$$

we conclude $[V, W] \in \Gamma(TM)$.
The lie bracket of hV and W is given by

$$\begin{aligned}
[hV, W](f) &= hV(W(f)) - W(hV(f)) = hV(W(f)) - W(h)V(f) - hW(V(f)) \\
&= h[V, W](f) - W(h)V(f),
\end{aligned}$$

and it is therefore not tensorial in V!.
To express the Lie bracket in local coordinates, pick a chart $\varphi : U \to \mathbb{R}^m$, express

$W = W^i \partial_{x_i}$ and $V = V^i \partial_{x_i}$ and compute the coefficients $[V,W]^i = [V,W](x^i)$

$$[V,W](x^k) = (V^j \partial_j W^i - W^j \partial_j V^i)\delta_{ik} = V^j \partial_j W^k - W^j \partial_j V^k \,.$$

Solution of 1.3: Let $(\rho_\alpha, U_\alpha)_\alpha$ be a locally finite partition of unity, that is $\mathrm{Supp}(\rho_\alpha) \subset U_\alpha$ and $\sum_\alpha \rho_\alpha(x) = 1$ only has finitely many non-zero elements for every fixed $x \in M$. Let $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ be a smooth atlas for the manifold M (notice that we can choose U_α the cover coming from the partition of unity by refining the partition). Define, for $v, w \in T_x M$, $x \in U_\alpha$,

$$g_{\alpha,x}(v,w) = \langle d\varphi_\alpha(x)[v], d\varphi_\alpha(x)[w] \rangle \,.$$

Recall that by definition, a Riemannian metric is a smooth (in x) positive definite symmetric bilinear form. The definition above, together with the fact that $d\varphi(x)$ is an invertible matrix for every x, provides a Riemannian metric on U_α. It remains to show that the g_α can be "glued" together using the partition of unity. Define for vector fields $V, W \in \Gamma(TM)$

$$g(V,W)(x) = \sum_\alpha g_\alpha(V(x), W(x))$$

Positive definiteness: $g(v,v)(x) = \sum_{\alpha=1}^{K_x} \rho_\alpha(x) g_\alpha(v,v) > 0$, since every g_α is positive definite and $\rho_\alpha > 0$ for the indices $\alpha = 1, ..., K_x$ (depending on x).
The biliearity as well as the smoothness carry over from the g_α given the smoothness of the partition of unity and the fact that the g_α appear linearly in the definition of g.

Solution of 1.4: We pick the local basis $\partial_r, \partial_\varphi$ of TM. By bilinearity it is enough to check that

$$g_2(dF(\partial_r), DF(\partial_\varphi)) = g_1(\partial_r, \partial_\varphi) = 0$$

$$g_2(dF(\partial_r), DF(\partial_r)) = g_1(\partial_r, \partial_r) = 1$$

$$g_2(dF(\partial_\varphi), DF(\partial_\varphi)) = g_1(\partial_\varphi, \partial_\varphi) = r^2$$

We compute

$$DF(\partial_r) = (\cos(\varphi), \sin(\varphi)) \qquad DF(\partial_\varphi) = (-r\sin(\varphi), r\cos(\varphi)),$$

the three identities to be verified follow.
Part 2.

Consider $w = A(z)$. Then since $ad - bc = 1$,

$$dw = \frac{dz}{(cz+d)^2}$$

and

$$\mathrm{Im}(w) = \frac{z - \overline{z}}{(cz+d)(c\overline{z}+d)}.$$

Therefore,

$$y^{-2}(dx^2 + dy^2) = -4\frac{dzd\overline{z}}{(z-\overline{z})^2} = -4\frac{dwd\overline{w}}{(w-\overline{w})^2}$$

and this shows that the Möbius transformations are local isometries. To prove that they are global it suffices to check that they are bijecive, with inverse given by $f^{-1}(z) = \frac{dz-b}{-cz+a}$. To prove that $\mathbb{H}^2$ is homogeneous, it is enough to prove that $f_t(z) = tz$ and $g_t(z) = z + t$ are isometries. Then, since composition of isometries is still an isometry, for any given two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{H}^2$, the map

$$A_{xy} := g_{y_1 - \frac{y_2 x_1}{x_2}} \circ f_{\frac{y_2}{x_2}}$$

is an isometry and satisfies $A(x) = y$. This proves that the hyperbolic plane is homogeneous.