10. Hadamard manifolds

10.1. Geodesics in Hadamard manifolds.

Let γ be an isometry of a Hadamard manifold (M, g). Show that $\mathrm{Min}(\gamma)$ is closed, geodesically convex (i.e. if $c\colon [0,1] \to \mathrm{Min}(\gamma)$ is a geodesic with $c(0), c(1) \in \mathrm{Min}(\gamma)$, then also $c(t) \in \mathrm{Min}(\gamma)$ for all $t \in [0,1]$), and γ-invariant.

10.2. "Uniqueness" and symmetries of hyperbolic space.

Prove that if M is a n-dimensional Riemannian manifold satisfying properties

1. for any given point all geodesic rays $x(t)$, $t \geq 0$ emanating from it are minimizing up to arbitrarily large values of $t > 0$ (note that this is stronger than geodesic completeness).
2. the sectional curvatures are constantly equal to -1,

and $p \in M$ then $\exp_p$ induces an isometry between $\mathbb{R}^n$ with metric

$$g(w, w) = \Big(w \cdot \tfrac{x}{|x|}\Big)^2 + \Big(|w|^2 - \Big(w \cdot \tfrac{x}{|x|}\Big)^2\Big)\frac{\sinh^2 |x|}{|x|^2} \tag{1}$$

and M. Deduce that given any two points p, q in the hyperbolic space $\mathbb{H}$ and any isometry H between their tangent spaces $T\mathbb{H}_p \to T\mathbb{H}_q$ there is a unique isometry $f : \mathbb{H} \to \mathbb{H}$ such that $f(p) = q$ and $df_p = H$.

10.3. Two dimensional Hadamard manifolds.

Let (M, g) be a two dimensional Hadamard manifold. For fixed point $p \in M$ and isometry $H : \mathbb{R}^2 \to TM_p$, consider $(\mathbb{R}^2, \overline{g})$ where $\overline{g} := (\exp_p \circ H)^* g$.

1. Show that $\overline{g}$ is of the form

$$\overline{g}_x(v, w) := \Big(v \cdot \tfrac{x}{|x|}\Big)\Big(w \cdot \tfrac{x}{|x|}\Big) + \frac{f^2(x)}{|x|^2}\Big(v \cdot w - \Big(v \cdot \tfrac{x}{|x|}\Big)\Big(w \cdot \tfrac{x}{|x|}\Big)\Big), \tag{2}$$

   where $f^2(x)/|x|^2$ is smooth (also at $x = 0$) and has limit 1 as $x \to 0$, and where $t \mapsto f(tx)$ is nonnegative and convex for any fixed $x \in \mathbb{R}^2 \setminus \{0\}$.

2. Reciprocally, show that $\mathbb{R}^2$ endowed with any metric $\overline{g}$ satisfying the properties established in 1. —and such that $g_x(v, w)$ extends to a smooth metric across $x = 0$— gives a model of a Hadamard manifold (simply connected with nonpositive sectional curvature at all points).

10.4. Asymptotic expansion of the circumference.
Let M be a manifold, $E \subset TM_p$ a linear 2-plane and $\gamma_r \subset E$ a circle with center 0 and radius $r > 0$ sufficiently small. Show that

$$L(\exp(\gamma_r)) = 2\pi \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right)$$

for $r \to 0$.

10. Solutions

Solution of 10.1:
Recall that $\mathrm{Min}(\gamma)$ was introduced in Definition 8.5 and it is the set of points thta minimizes the displacement function for the given isometry γ:

$$\mathrm{Min}(\gamma) = \{p \in M \mid d_\gamma(p) = \mathrm{dist}(p, \gamma(p)) = \inf_{q \in M} d_\gamma(q) \equiv |\gamma|.$$

Let us start by showing that it is closed:
Assume $p_k \in \mathrm{Min}(\gamma)$ is a sequence of points converging to $p_k \to p \in M$. Then

$$d_\gamma(p) = d_\gamma(p_k) \equiv |\gamma| \implies p \in \mathrm{Min}(\gamma).$$

Next we prove γ-invariance:
Let $p \in \mathrm{Min}(\gamma)$. Then, since isometries preserve distance,

$$d_\gamma(\gamma(p)) = \mathrm{dist}(\gamma(p), \gamma(\gamma(p))) = \mathrm{dist}(p.\gamma(p)) = |\gamma| \implies \gamma(p) \in \mathrm{Min}(\gamma).$$

It remains to show convexity:
Let c be a geodesic with $c(0), c(1) \in \mathrm{Min}(\gamma)$. Then $\gamma(c(t)) =: d(t)$ is also a geodesic. By Lemma 8.10, $g(t) := \mathrm{dist}(c(t), d(t)))$ is convex, and by assumption on $c(0), c(1)$, $g(0) = g(1) = |\gamma|$, which implies $g(t) \equiv |\gamma|$ and hence $c(t) \in \mathrm{Min}(\gamma)$ for $0 \leq t \leq 1$.

Solution of 10.2: Let $c(t)$ be a geodesic on M and $Y(t)$ a Jacobi field. Take $E(t)$ parallel and orthogonal to $\dot{c}(t)$. Then, since M has sectional curvatures constantly equal to -1, $Y = fE$ satisfies the Jacobi field equation provided $f'' - f = 0$, which has solutions cosh and sinh.

Notice that $t \mapsto (\exp_p)((v + w)t)$ gives a geodesic for all , for all fixed $v, w \in TM_p$. Hence, the variation $Y(t) = d(exp_p)_{vt}(wt) = td(exp_p)_{vt}(w)$ is a Jacobi field. Hence As shown in the lecture, this fact and Gauss' lemma allows us to compute $|d(\exp_p)_v(w)|$ as

$$|d(\exp_p)_v(w)|^2 = (w \cdot \tfrac{v}{v})^2 + \Big(|w|^2 - \big(w \cdot \tfrac{v}{|v|}\big)^2\Big)\frac{\sinh^2|v|}{|v|^2}.$$

In other words the metric of M in normal coordinates x is given by (1).

Also, since by assumption M satisfies the property c) in the previous excercise we obtain that the map $\exp_p : TM_p \to M$ is injective (and a diffeomorphism). It follows that M is isometric to $\mathbb{R}^n$ with metric g given by (1).

Finally, since we can replace p by any other point q and the expression of g in local coordinate given by $\exp_q$ will be the same, and since the metric is clearly rotationally invariant, it follows that for any isometry between $T\mathbb{H}_p \to T\mathbb{H}_q$ there is a unique isometry $f : \mathbb{H} \to \mathbb{H}$ such that $f(p) = q$ and $df_p = H$ (with is given by $(\exp_q) \circ H \circ (\exp_p)^{-1}$.

Solution of 10.3:(a) Fix $x \in \mathbb{R}^2 \setminus 0$ and let $c_x(t)$ be a geodesic emanating from p, with unit initial velocity $H(x)/|x| \in TM_p$. Let $E(t)$ be a parallel unit vector field along c_x which is orthogonal to $c_x'(t)$. Notice that $Y := \phi(t)E$ is a Jacobi field if, and only if, $\phi'' + (K \circ c_x)\phi = 0$. Now, for fixed $x \neq 0$ and let $w \in \mathbb{R}^2$ be perpendicular to x, we have

$$Y(t) := d(\exp_p \circ H)_{tx/|x|}(tw)$$

is a Jacobi field satisfying $Y(0) = 0$ and $Y'(0) = w$. Hence, by definition pullback metric $\overline{g} := (\exp_p \circ H)^* g$, for $t > 0$ we have

$$\begin{aligned}\overline{g}_{tx/|x|}(w,w) &= g\Big(d(\exp_p \circ H)_{tx/|x|}(w), d(\exp_p \circ H)_{tx/|x|}(w)\Big) \\ &= t^{-2} g\Big(Y(t), Y(t)\Big) = t^{-2} g\Big(|w|\phi E(t), |w|\phi E(t)\Big) = (\phi/t)^2|w|^2,\end{aligned}$$

where ϕ is the unique solution of $\phi'' + (K \circ c_x)\phi = 0$ with initial condititiion $\phi(0) = 0$ and $\phi'(0) = 1$.

Hence, setting $t = |x|$ in the equation above and defining $f(x) = \phi(|x|)$ as the unique solution $\phi'' + (K \circ c_x)\phi = 0$ with $\phi(0) = 0$ and $\phi'(0) = 1$ evaluated at time $t = |x|$, we obtain

$$\overline{g}_x(w,w) = (f(x)/|x|)^2|w|^2,$$

Using that from Gauss' lemma

$$\overline{g}_x(v,w) = v \cdot w$$

whenever v parallel to x (and w is any vector), we obtain (2).

Finally observer that $\phi'' = -(K \circ c_x)\phi \geq 0$ implies that ϕ is convex (and hence so is $t \mapsto f(tx)$). Also, by l'Hopital's rule, $\lim_{t\to 0} \phi(t)/t = \phi'(0) = 1$ and hence the limit of $f^2(x)/|x|^2 \to 1$ as $x \to 0$.

(b) Consider now $\mathbb{R}^2$ endowed with a metric of the form (2). Take polar corrdinates (r, θ) in $\mathbb{R}^2 \setminus 0$. Notice that coordinates the metric is of the form

$$(g_{i,j}) = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$$

where $E = E(r, \theta) = \overline{g}(\partial_\theta, \partial_\theta) = f^2(r\cos\theta, r\sin\theta)$.

The condition that f is convex along rays from 0 reads $\left(\sqrt{E}\right)_{11} \geq 0$.

In order to compute the curvature, let us compute the Chistoffel symbols (we still use polar coordinates). The only nonzero ones are:

$$\Gamma^2_{22} = \frac{E_2}{2E}, \quad \Gamma^2_{12} = \Gamma^2_{12} = \frac{E_1}{2E}, \quad \Gamma^1_{22} = \frac{-E_1}{2}$$

Hence, direct computation shows:

$$K = \frac{E_1^2}{4E^2} - \frac{E_{11}}{2E} = -\frac{\left(\sqrt{E}\right)_{11}}{\sqrt{E}} \leq 0.$$

Solution of 10.4: Let $v, w \in TM_p$ be an orthonormal basis of E. Then the circle can be parametrized by $\gamma_r(\varphi) = r(v\cos\varphi + w\sin\varphi)$. For some fixed $\varphi_0 \in [0, 2\pi]$, consider the Jacobi field $Y_{\varphi_0}(r)$ associated to the geodesic variation $V(\varphi, r) := \exp(\gamma_r(\varphi))$ of the geodesic $c_{\varphi_0}(r) := \exp(\gamma_r(\varphi_0))$. Then it holds

$$L(\exp(\gamma_r)) = \int_0^{2\pi} |Y_\varphi(r)|\, d\varphi.$$

We will now compute the Taylor expansion for $|Y_0(r)|$, all other cases are similar. We have $Y_0(0) = 0$ and $Y_0'(0) = w$. From the Jacobi equation we also get

$$Y_0''(0) = -R\left(Y_0, c_0'\right) c_0'\Big|_{r=0} = 0.$$

Now taking the derivative of the Jacobi equation, we get

$$Y_0'''(0) = -\frac{D}{dr} R\left(Y_0, c_0'\right) c_0'\Big|_{r=0} = -R\left(Y_0', c_0'\right) c_0'\Big|_{r=0} = -R(w,v)v.$$

It follows that

$$|Y_0(r)| = r - \frac{R(w,v,w,v)}{6} r^3 + \mathcal{O}(r^4).$$

Therefore, we finally get

$$L(\exp(\gamma_r)) = \int_0^{2\pi} \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4)\right) d\varphi = 2\pi \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4)\right),$$

as it was to show.