11. Differential forms, part one

This sheet contains the first exercises involving differential forms. You can solve the exercises using material covered before chapter 9.3 in the lecture notes.

11.1. Characterization of orientability.

Let M be a smooth m-dimensional manifold. Show that M is orientable if and only if there exists a nowhere vanishing m-form on M.

11.2. Properties of the wedge product.

Prove the following two statements (see Remark 9.3 in the lecture notes): given $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^l V^*$,

1. $(\omega \wedge \eta) \wedge \rho = \omega \wedge (\eta \wedge \rho)$
2. $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.

11.3. Example of pull-back.

Let $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{0,0\})$ be the 1-form given by

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

and let f be the map $f(r,\theta) = (r\cos(\theta), r\sin(\theta))$ defined on $\mathbb{R}^2 \setminus (-\infty, 0]$.
Compute $f^*\omega$.

11.4. Invariant n-forms on Lie groups.

Prove that the space of left (or right) invariant n-forms on a Lie group (G, g) of dimension n is a one dimensional vector space.

11. Solutions

Solution of 11.1:

In example 9.9.ii. we saw that the form

$$\omega = \sqrt{\det(g(\partial_{x_i}, \partial_{x_j}))} dx^1 \wedge \dots \wedge dx^n \in \Omega^n(\varphi(U))$$

is well-defined for a positively oriented chart $: U \to M$ and independent of the choice of positively oriented chart. It follows that for an oriented manifold M (which admits a positively oriented atlas), $\omega \in \Omega^n(M)$ is a globally well-defined non-vanishing n-form. Conversely, assume such a form ω exists. Consider two overlapping charts defined on an open set, $\psi, \varphi : U \to M$. We need to show that the charts can be chosen so that

$$\det(\partial_{x_i} y^j) > 0,$$

where x^i and y^i are the local coordinates associated to φ and ψ respectively. Since for a manifold of dimension n we have $\dim(\Lambda^n(T_pM)) = 1$,

$$\omega = h(x)dx^1 \wedge ... \wedge dx^n = g(x)dy^1 \wedge ...dy^n$$

and up to switching two indices let us assume that $g, h > 0$. Then, by the same computation as in 9.9.ii we can express the form in the two different charts as follows:

$$dy^1 \wedge ...dy^n \det\left(\frac{\partial \psi}{\partial x}\right) dx^1 \wedge ... \wedge dx^n$$

from which we conclude

$$\det\left(\frac{\partial \psi}{\partial x}\right) > 0.$$

Solution of 11.2:

Both statements follow directly from the definition and the properties of permutations.

Solution of 11.3:

$$dx = \cos(\theta)dr - r\sin(\theta)d\theta$$

$$dy = \sin(\theta)dr + r\cos(\theta)d\theta$$

Inserting these two equations in the definitions above gives:

$$f^*\omega = -\frac{r\sin(\theta)}{r^2}(\cos(\theta)dr - r\sin(\theta)d\theta) + \frac{r\cos(\theta)}{r^2}(\sin(\theta)dr + r\cos(\theta)d\theta) = d\theta.$$

Solution of 11.4:

Let $\omega \in \Omega^n(G)$ be left-invariant. By definition this means that for the left multiplication map $l_a : G \to G$, $l_a(b) = ab$,

$$l_a^*\omega = \omega.$$

In particular, if two left-invariant forms agree at a point, they coincide on the entire group G:

$$\omega_e = \widetilde{\omega}_e \implies \omega = \widetilde{\omega}.$$

The result then follows from the fact that the space $\Lambda^n(T_e G)$ has dimension 1.