12. Differential forms, part two

12.1. Application of Stokes' theorem.

Let M be a smooth oriented manifold. Write $\omega \in \Omega^m(M)$ for the volume form of a Riemannian metric g on M. Let $V \in \Gamma(TM)$.

1. Show that $d(i_V \omega) = \mathrm{d}iv_g(V)\omega$.
2. Write h for the metric on ∂M induced by g via pullback along the inclusion map $i\colon \partial M \hookrightarrow M$, and write $\sigma \in \Omega^{m-1}(\partial M)$ for the volume form of $(\partial M, h)$. Show that $\sigma = i_N\omega$ where $N \in \mathcal{C}^\infty(\partial M; T_{\partial M}M)$ is an outward pointing unit normal vector field (i.e. $g(N,N) = 1$ and $N \perp T\partial M$). Prove also the existence of such an N.
3. Prove the *divergence theorem*: $\int_M \mathrm{d}iv_g(V)\omega = \int_{\partial M} g(V,N)\sigma$.

12.2. Poincare Lemma.

The goal of this exercise is to prove the Poincaré lemma for compactly supported cohomology in the following form: for a smooth manifold M, we have $H_c^{k+1}(\mathbb{R} \times M) \cong H_c^k(M)$.

1. Define the map $\pi_*\colon \Omega^{k+1}(\mathbb{R} \times M) \to \Omega^k(M)$, given by *integration on the fibers*, as follows: for $\omega = \mathrm{d}t \wedge \omega_0 + \omega_1$ with $\omega_j = \sum_I \omega_{j,I}(t,x)\mathrm{d}x^I$, $j = 0,1$, where the $\omega_{j,I}$ have compact support in t, set
$$\pi_*\omega := \int_{-\infty}^{\infty} \omega_0(t)\,\mathrm{d}t := \sum_I \left(\int_{-\infty}^{\infty} \omega_{0,I}(t,x)\,\mathrm{d}t\right)\mathrm{d}x^I.$$
Show that $\pi_*\mathrm{d} = \mathrm{d}\pi_*$. Therefore, π_* induces a map in cohomology which we denote
$$\pi_*\colon H^{*+1}(\mathbb{R} \times M) \to H^*(M).$$
2. Let $e = e(t)\,\mathrm{d}t$ where $e \in \mathcal{C}_c^\infty(\mathbb{R})$ has total integral 1. Define the map
$$e_*\colon \Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R}), \qquad \phi \mapsto \phi \wedge e.$$
Show that e_* induces a map in cohomology which we denote also by
$$e_*\colon H^*(M) \to H^{*+1}(\mathbb{R} \times M).$$
3. Show that $\pi_* \circ e_* = \mathrm{I}d$ on $H^*(M)$.

4. Show that $e_* \circ \pi_* = \mathrm{I}d$ on $H^*(\mathbb{R}^\times M)$ as follows. Define the map

$$\begin{aligned} &K\colon \Omega^*_{\mathrm{c}}(\mathbb{R}\times M) \to \Omega^{*-1}_{\mathrm{c}}(\mathbb{R}\times M), \\ &\quad K(f\pi^*\phi) := 0 \quad (f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}\times M),\ \phi \in \Omega^k_{\mathrm{c}}(M)), \\ &\quad K(f\pi^*\phi \wedge \mathrm{d}t)(t) = (\pi^*\phi)\int_{-\infty}^t f - \phi A(t) \int_{-\infty}^{\infty} f \end{aligned}$$

where $A(t) = \int_{-\infty}^t e$. Verify that

$$1 - e_*\pi_* = (-1)^k(\mathrm{d}K - K\mathrm{d}) \quad \text{on} \quad \Omega^{k+1}_{\mathrm{c}}(\mathbb{R}\times M)$$

and use this to conclude the argument.

12.3. Cohomology of the sphere.
The goal of this exercise is to show that the cohomology groups of $\mathbb{S}^m$ are given by $H^k(\mathbb{S}^m) \cong \mathbb{R}$ for $k = 0, m$, and 0 for all other k. Proceed by induction on m using the Mayer–Vietoris sequence. (The case $m = 1$ may be assumed, as it was discussed in class.)

12.4. Metrics of negative sectional curvature.
The goal of this exercise is to prove the following theorem: Let M and N be compact, connected, smooth manifolds of positive dimension. Then $M \times N$ does not admit a metric of negative sectional curvature.

Proceed as follows:

1. Show that at least one of the factors M or N is simply connected.
2. Assume that M is simply connected. Show that the universal covering space of $M \times N$ is $M \times \widetilde{N} \cong \mathbb{R}^{m+n}$ where $\widetilde{N}$ is the universal covering space of N.
3. Show that $\mathbb{R}^{m+n}$ cannot be diffeomorphic to $M \times \widetilde{N}$ by considering an m-form $\omega \in \Omega^m(M)$ with $\int_M \omega = 0$ and computing $\int_{s(M)} \pi^*\omega$, where $s\colon M \to M \times \widetilde{N}$, $s(p) = (p, q_0)$ (for some fixed $q_0 \in \widetilde{N}$) and $\pi\colon M \times N \to M$ is the projection, in two different ways. (Hint: what is $\mathrm{d}\pi^*\omega$?)

12. Solutions

Solution of 12.1:

1.

$$\begin{aligned} d(i_V\omega) &= d\left(i_V\left(dx^1\wedge\cdots\wedge dx^n\right)\right) \\ &= d\left(\sum_{i=1}^n (-1)^{i+1}V_i\; dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n\right) \\ &= \sum_{i=1}^n \frac{\partial V_i}{\partial x^i}\; dx^1\wedge\cdots\wedge dx^n \\ &= (\operatorname{div} V)\omega, \end{aligned}$$

2. see Lemma 9.17.ii, discussed in the lectures.

3.

$$i_V\omega = \omega(\langle V,N\rangle N + (V-\langle V,N\rangle N),\cdots) = \omega(\langle V,N\rangle N,\cdots) = \langle V,N\rangle\sigma,$$

to conclude it then suffices to apply Stokes theorem:

$$\int_M div_g(V)\omega = \int_M d(i_V\omega)) = \int_{\partial M} i_v\omega = \int_{\partial M} g(V,N)\sigma\,.$$

Solution of 12.2: See pg. 37-39 of the book "Differential forms in algebraic topology".

Solution of 12.3:

Let

$$A = \{(x_0,...,x_n \in S^n \mid x_0 > -1/2)\}$$

$$A = \{(x_0,...,x_n \in S^n \mid x_0 < 1/2)\}$$

A and B are open and their union is S^n. The intersection has the homotopy type of S^{n-1}, while both A and B are contractible. Inserting this information in the Meyer-Vietoris long exact sequence,

$$\cdots\to H^m(A)\oplus H^m(B)\to H^m(A\cap B)\to H^{m+1}(S^n)\to H^{m+1}(A)\oplus H^{m+1}(B)\to\cdots$$

we obtain isomorphisms

$$0\to H^m(S^{n-1})\to H^{m+1}(S^n)\to 0$$

for $n,m>0$.

Solution of 12.4: As suggested by the hints, we argue by contradiction. Let us assume that $M \times N$ carries a metric of negative sectional curvature.

1. Assume both M and N are not simply connected. Then $\pi_1(M \times N) \cong \pi_1(M) \times \pi_1(N)$ contains a subgroup isomorphic to $\mathbb{Z}^2$. This contradicts Preissmann's Theorem 8.12 and we can therefore assume that M is simply connected.

2. For the universal cover $M \times \widetilde{N}$ of $M \times N$ it must hold that $M \times \widetilde{N}$ is contractible and moreover diffeomorphic to $\mathbb{R}^{m+n}$ by Cartan-Hadamard's Theorem 8.1.

3. Since M is assumed to be contractible, it must be orientable and $H^m(M) \neq 0$. But $0 \equiv \pi_* : H^k(M \times \widetilde{N}) \to H^k(M)$, which contradicts $\pi^* \circ i^* = id$