13. Cohomology

13.1. Cohomology of punctured plane.

Compute $H^1(\mathbb{R}^2\backslash\{(x_1,y_1)\cup(x_2,y_2)\})$ and provide an explicit description of its generators.

13.2. Compactly supported cohomology.

Prove Lemma 9.30 in the lecture notes.

13.3. Non-compact manifolds.

Prove that for any non-compact orientable manifold M of dimension m, $H^m_{dR}(M)=0$

13.4. Homeomorphic manifolds.

Assuming that two smooth manifolds M and N are diffeomorphic, show that $H^k_c(M)\cong H^k_c(N)$.

13. Solutions

Solution of 13.1: *Sketch.* $M:=\mathbb{R}^2\backslash\{p_1\cup p_2\}$ can be covered by two open sets which have as intersection an infinite rectangle. The long exact Meyer-Vietoris sequence becomes,

$$0\to\mathbb{R}\to\mathbb{R}^2\to\mathbb{R}\to H^1(M)\to\mathbb{R}^2\to 0\to H^2(M)\to 0.$$

Since the sequence is exact, $H^1(M)$ must be at least of dimension two and at most of dimension three. But in order for the sequence to be exact one can check that it must hold $H^2(M)\cong\mathbb{R}^2$.

Solution of 13.2:

Solution of 13.3: See Lee's "Introduction to smooth manifolds" Theorem 17.32.

Solution of 13.4: This is a consequence of 9.13 and 9.22.iii).