# 2. Length and volume, Levi-Civita connection, vector fields

## 2.1. Parametrizations, lengths and volumes.

As discussed in the lectures, for a smooth curve $\gamma : [a,b] \to M$, into a Riemannian manifold $(M,g)$, one defines its length by

$$L(\gamma) = \int_a^b g(\gamma'(t), \gamma'(t))^{1/2} dt.$$

We also define its energy

$$E(\gamma) = \int_a^b g(\gamma'(t), \gamma'(t)) dt.$$

1. Prove that $L(\gamma)$ is independent of the parametrization: for any diffeomorphism $\psi : [c,d] \to [a,b]$, $L(\gamma \circ \psi) = L(\gamma)$.
2. Prove that in general $E(\gamma \circ \psi) \neq E(\gamma)$. Give sufficient conditions on $\psi$, such that equality holds.
3. Let $\mathcal{A}$ be the family of diffeomorphisms between $[a,b]$ and $[c,d]$ for arbitrary $c \neq d$. Prove that

$$\sup_{\psi \in \mathcal{A}} E(\gamma \circ \psi) = \infty, \qquad \inf_{\psi \in \mathcal{A}} E(\gamma \circ \psi) = 0$$

4. Bonus question, to be discussed later: what happens if the length of $[c,d]$ is prescribed?

## 2.2. Connections.

Let $M$ be an $m$-dimensional smooth manifold. Suppose for all $V, W \in \Gamma(TM)$ we are given $D_V W$ with the following properties for all $f \in \mathcal{C}^\infty(M)$, $V, W \in \Gamma(TM)$:

$$D_{fV} W = f D_V W, \qquad D_V(fW) = (Vf)W + f D_V W.$$

$$D_{V_1+V_2} W = D_{V_1} W + D_{V_2} W \quad D_V(W_1 + W_2) = D_V W_1 + D_V W_2 \,.$$

One calls $D$ a **connection**.

1. Show that in local coordinates $x = (x^1, \dots, x^m)$ on $M$ there exist smooth functions $\Gamma_{ij}^k$ (called **connection coefficients**, $i,j,k = 1, \dots, m$, so that $D_V W = V^i(\partial_{x^i} W^j)\partial_{x^j} + V^i W^j \Gamma_{ij}^k \partial_{x^k}$.

2. Show, conversely, that this formula defines a map (in the local coordinate chart) satisfying the properties above.

3. Show that $D$ is **torsion-free**, meaning $D_V W - D_W V = [V,W]$ for all $V, W \in \Gamma(TM)$, if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$.

4. Show that there exists a connection $D$.

5. Fix a connection $D_0$. Prove that $\{D - D_0 \colon D \text{ is a connection}\} \cong \Gamma(T_{1,2}M)$ via $D \mapsto \Big((V,W) \mapsto D_V W - (D_0)_V W\Big)$. Thus, the space of connections is an infinite-dimensional affine space modelled on $\Gamma(T_{1,2}M)$.

6. State (and prove) an analogous result for connections on a vector bundle $E \to M$.

### 2.3. Levi-Civita connection of immersed submanifold.

1. Let $(\overline{M}, \overline{g})$ be a Riemannian manifold with Levi-Civita connection $\overline{D}$, and let M be a submanifold of $\overline{M}$ , equipped with the induced metric $g := i^*\overline{g}$, where $i : M \to \overline{M}$ is the inclusion map. Show that the Levi-Civita connection $D$ of $(M, g)$ satisfies $D_X Y = (\overline{D}_X Y)^T$ for all $X, Y \in \Gamma(TM)$, where the superscript $T$ denotes the component tangential to $M$ and $\overline{D}_X Y$ is <u>defined</u> as $\overline{D}_X Y := \overline{D}_{\overline{X}} \overline{Y}$ for any extensions $\overline{X}, \overline{Y} \in \Gamma(TM)$ of $X, Y$.

2. Let $(M, g)$ be a smooth manifold with Levi-Civita connection $D$. Consider the metric $\tilde{g} = \varphi g$ for a positive smooth function $\varphi : M \to \mathbb{R}_{>0}$. Compute the Levi-Civita connection of $(M, \tilde{g})$. What happens for $\varphi \equiv c > 0$? How do you explain it?
   *Hint:* By problem 2.2. any two connections differ by a tensor, try to compute that tensor to obtain the result.

### 2.4. Pullbacks.

Let $N$ be a smooth manifold, and let $(M, g)$ be a Riemannian manifold. Let $F \colon N \to M$ be a smooth map.

1. Let $V \colon N \to TM$ be a vector field along $F$. Let $p \in N$. Show that there exist a neighborhood $U \subset N$ of $p$, smooth vector fields $V_1, \ldots, V_m \in \Gamma(TM)$, and smooth functions $f^1, \ldots, f^m \in \mathcal{C}^\infty(N)$ so that $V = f^i F^*(V_i)$ on $U$. (Here $F^*(V_i) \colon N \to TM$ is the vector field $N \ni q \mapsto V(F(q)) \in T_{F(q)}M$ along $F$.)

2. Let $V, W : N \to TM$ be vector fields along $F$ and $Z \in \Gamma(TN)$. Write $\langle V, W\rangle(p) = g_{F(p)}(V_p, W_p)$. Show that $Z\langle V, W\rangle = \langle \nabla_Z V, W\rangle + \langle V, \nabla_Z W\rangle$.
   *Hint.* Show this first in the case that $V, W$ are pullbacks along $F$ of smooth vector fields on $M$. Conclude in the general case using the first part.

## 2. Solutions

### Solution of 2.1:

1. Let $\psi : [c, d] \to [a, b]$ be a diffeomorphism. Then, $\gamma \circ \psi : [c, d] \to M$ is a smooth curve and $(\gamma \circ \psi)'(t) = \gamma'(\psi(t))\psi'(t)$. By definition of $L$,

$$\begin{aligned} L(\gamma \circ \psi) &= \int_c^d g(\gamma'(\psi(t))\psi'(t), \gamma'(\psi(t))\psi'(t))^{1/2} dt \\ &= \int_c^d \psi'(t) g(\gamma'(\psi(t)), \gamma'(\psi(t)))^{1/2} dt \\ &= \int_a^b g(\gamma'(\psi(t)), \gamma'(\psi(t)))^{1/2} dt = L(\gamma)\,. \end{aligned}$$

2. As for part 1,

$$\begin{aligned} E(\gamma \circ \psi) &= \int_c^d g(\gamma'(\psi(t))\psi'(t), \gamma'(\psi(t))\psi'(t)) dt \\ &= \int_c^d \psi'(t)^2 g(\gamma'(\psi(t)), \gamma'(\psi(t))) dt \\ &= \int_a^b g(\gamma'(\psi(t)), \gamma'(\psi(t)))\psi'(t) dt\,. \end{aligned}$$

   In particular $E(\gamma \circ \psi) = E(\gamma)$ if and only if the reparametrization $\psi$ is by constant unit speed $\psi' \equiv 1$.

3. Consider the diffeomorphism $\lambda_t : [0, t] \to [0, 1]$, $\lambda_t(x) = t^{-1}x$. By the computation in part 2. above,

$$E(\gamma \circ \lambda_t) = t^{-1} E(\gamma).$$

   Letting $t \to 0$ and $t \to \infty$, the claim follows.

### Solution of 2.2:

1. Recall the vector fields associated with the coordinate system $(x^1, ..., x^m)$ on $U \subset M$, $\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}\right) = (X^1, ..., X^m)$, $X^i \in \Gamma(TU)$. Define, $\Gamma^k_{ij} : U \to \mathbb{R}$ by

$$\Gamma^k_{ij}(p)X^k(p) = \nabla_{X_i} X_j(p)\,.$$

This determines the function $\Gamma^k_{ij}$ uniquely on $U$. Using the local basis of $TM|_U$ given by $X^1, .., X^m$, write

$$V = V^{Xi} \quad W = W^j X^j\,.$$

Then,

$$\begin{aligned}\nabla_V W = \nabla_{V^i X^i}(W^j X^j) = V^i \nabla_{X^i}(W^j X^j) = V^i W^j \nabla_{X^i} X^j + V^i X^i(W^j) X^j \\ = V^i W^j \Gamma^k_{ij} X^k + V^i X^i(W^j) X^j\end{aligned}$$

2. Given the formula

$$V^i(X^i(W^j))X^j + V^i W^j \Gamma^k_{ij} X^k\,,$$

we verify:

$$D_{fV}(W) = fV^i(X^i(W^j))X^j + fV^i W^j \Gamma^k_{ij} X^k = f(V^i(X^i(W^j))X^j + V^i W^j \Gamma^k_{ij} X^k) = fD_V W\,,$$

$$\begin{aligned}D_V(fW) &= V^i(X^i(fW^j))X^j + V^i f W^j \Gamma^k_{ij} X^k \\ &= V^i f(X^i(W^j))X^j + V^i W^j (X^i(f))X^j + V^i f W^j \Gamma^k_{ij} X^k \\ &= fD_V W + V^i W^j (X^i(f))X^j = fD_V W + V(f)W\end{aligned}$$

3. By direct computation,

$$D_V W - D_W V = V^i X^i(W^j) X^j + V^i W^j \Gamma^k_{ij} X^k - W^i X^i(V^j) X^j - W^i V^j \Gamma^k_{ij} X^k$$

$$[V, W] = (V^i X^i(W^j) - W^i X^i(V^j))X^j$$

the second equation was proven in exercise 1.2. Taking the difference,

$$D_V W - D_W V - [V, W] = V^i W^j \Gamma^k_{ij} X^k - W^i V^j \Gamma^k_{ij} X^k$$

which vanishes if and only if it does so component wise, i.e. if and only if

$$V^i W^j \Gamma_{ij}^k - W^i V^j \Gamma_{ij}^k = 0 \quad \text{for all } V, W \in \Gamma(TU)\,. \tag{1}$$

It is thus clear that $\Gamma_{ij}^k = \Gamma_{ji}^k$ is a sufficient condition for $D$ to be torsion free. To see that it is necessary, choose $V = X^i$, $W = X^j$, then (1) becomes

$$\Gamma_{ij}^k = \Gamma_{ji}^k\,,$$

which concludes the proof of the claim.

4. To prove existence of a connection $D$ we show that it is possible to patch together the connections $D$ defined locally on a coordinate chart in part 1 using a partition of unity. Let $(U_\alpha, \rho_\alpha)$ be a locally finite partition of unity such that $U_\alpha$ is the domain of a chart $\varphi_\alpha$ (all compatible). Then, for arbitrary vector fields $V, W \in \Gamma(TM)$ define

$$D_V W = \sum_\alpha \rho_\alpha D_V(W)$$

We check property 2:

$$D_{fV} W = \sum_\alpha \rho_\alpha (f D_V W + V(f) W) = f D_V W + V(f) W\,.$$

5. Introducing the notation

$$A(V, W) = D_V W - (D_0)_V W\,,$$

$$A(fV, W) = D_{fV} W - (D_0)_{fV} W = f A(V, W)$$

$$A(V, fW) = D_V(fW) - (D_0)_V(fW) = f A(V, W) + V(f) W - V(f) W = f A(V, W)$$

and $A(V_1 + V_2, W) = A(V_1, W) + A(V_2)$, $A(V, W_1 + W_2) = A(V, W_1) + A(V, W_2)$ follow similarly using the properties of connections.

**Solution of 2.3:**

1. As we have seen in the lecture, that $(\bar{D}_{\bar{X}} \bar{Y})_p$ only depends on $\bar{X}_p$ and $\bar{Y} \circ c$, where $c\colon (-\epsilon, \epsilon) \to \bar{M}$ is a curve with $\dot{c}(0) = \bar{X}$. Hence $\bar{D}_X Y$ is independent of the choice of the extensions $\bar{X}$ and $\bar{Y}$.

Clearly, $(\bar{D}_X Y)^{\mathrm{T}}$ defines a linear connection. It remains to prove that this connection is compatible with $g$ and torsion-free. For $X, Y, Z \in TM$, we have

$$\begin{aligned} Zg(X,Y) &= \bar{Z}\bar{g}(\bar{X},\bar{Y}) = \bar{g}(\bar{D}_{\bar{Z}}\bar{X},\bar{Y}) + \bar{g}(\bar{X},\bar{D}_{\bar{Z}}\bar{Y}) \\ &= \bar{g}((\bar{D}_Z X)^{\mathrm{T}},\bar{Y}) + \bar{g}(\bar{X},(\bar{D}_Z Y)^{\mathrm{T}}) = g((\bar{D}_Z X)^{\mathrm{T}},Y) + g(X,(\bar{D}_Z Y)^{\mathrm{T}}) \end{aligned}$$

and

$$(\bar{D}_X Y) - (\bar{D}_Y X) = (\bar{D}_{\bar{X}}\bar{Y})^{\mathrm{T}} - (\bar{D}_{\bar{Y}}\bar{X})^{\mathrm{T}} = [\bar{X},\bar{Y}]^{\mathrm{T}} = [X,Y].$$

2. Let $D$ be the Levi-Civita connection of $(M,g)$. Notice that in problem 2.2 we did not need a metric on $M$ to verify the properties of a connection, hence $D$ is also a connection on $(M,\tilde{g})$ for the Riemannian metric $\varphi g = \tilde{g}$. The Levi Civita connection needs to satisfy two properties: the torsion free condition and the metric compatibility (see Def. 3.5). Only the second one depends on the metric. Let $\tilde{\nabla}$ denote the Levi-Civita connection of $(M,\tilde{g})$. Let $A = \nabla - \tilde{\nabla}$. Recall that the Levi-Civita connection is uniquely determined by the Koszul formula (see Theorem 3.7),

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad - g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned}$$

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X(\tilde{g}(Y,Z)) + Y(\tilde{g}(X,Z)) - Z(\tilde{g}(X,Y)) \\ &\quad - \tilde{g}([Y,X],Z) - \tilde{g}([X,Z],Y) - \tilde{g}([Y,Z],X) \\ &= X(\varphi)g(Y,Z) + \varphi X(g(Y,Z)) + Y(\varphi)g(X,Z) + \varphi Y(g(X,Z)) \\ &\quad - Z(\varphi)g(X,Y) - \varphi Z(g(X,Y)) \\ &\quad - \varphi g([Y,X],Z) - \varphi g([X,Z],Y) - \varphi g([Y,Z],X) \\ &= 2\varphi g(\nabla_X Y, Z) + X(\varphi)g(Y,Z) + Y(\varphi)g(X,Z) - Z(\varphi)g(X,Y)\,. \end{aligned}$$

We obtained,

$$2\tilde{g}(\nabla_X Y - A(X,Y), Z) = 2\tilde{g}(\nabla_X Y, Z) + X(\varphi)g(Y,Z) + Y(\varphi)g(X,Z) - Z(\varphi)g(X,Y)$$

which implies

$$-2A(X,Y) = \varphi^{-1}(X(\varphi)Y + Y(\varphi)X - \mathrm{grad}(\varphi)g(X,Y))$$

In particular,

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{\varphi^{-1}}{2}(X(\varphi)Y + Y(\varphi)X - \text{grad}(\varphi)g(X,Y))$$

Remark: writing $\varphi = e^{2f}$, the formula simplifies to

$$\tilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - \text{grad}(f)g(X,Y)\,.$$

**Solution of 2.4:** Given $p \in N$, $F(p) \in M$. Let $(x^1, ..., x^m)$ be a chart on $U \subset M$ around $F(p)$. Consider the associated vector fields $\partial_1, ..., \partial_m$ which locally give a frame of $TM$. Then, $\pi \circ V = F$ is continuous ($\pi$ denotes the projection $TM \to M$), and on the open set $O = (\pi \circ V)^{-1}(U)$,

$$V(p) = f^i(p)\partial_i(F(p)) =: f^i(p)F^*\partial_i(p)\,.$$

Part 2. By Definition 3.10, for $W(q) = W^k\partial_k(F(q))$

$$\nabla_Z W(p) = (Z_p(W^k) + d_pF^i(Z)W^j(p)\Gamma^k_{ij}(F(p)))\partial_k|_{F(p)}$$

and hence,

$$\begin{aligned}\langle \nabla_Z V, W\rangle + \langle V, \nabla_Z W\rangle =&(Z_p(V^k) + d_pF^i(Z)V^j(p)\Gamma^k_{ij}(F(p)))W^l g_{kl}(F(p)) \\ &+ (Z_p(W^k) + d_pF^i(Z)W^j(p)\Gamma^k_{ij}(F(p)))V^l g_{kl}(F(p))\end{aligned}$$

$$\begin{aligned}Z\langle V, W\rangle &= Z\langle V^iF^*\partial_i, W^j\partial_j\rangle = Z(V^iW^jg_{F(p)}(\partial_i(F(p), \partial_j(F(p)))) \\ &= Z(V^iW^jg_{ij}(F(p))) \\ &= Z(V^i)W^jg_{ij}(F(p)) + Z(W^j)V^ig_{ij}(F(p)) + V^iW^jZ(g_{ij}(F(p)))\,.\end{aligned}$$

To conclude it suffices to check:

$$d_pF^i(Z)V^j(p)\Gamma^k_{ij}(F(p)))W^lg_{kl}(F(p) + d_pF^i(Z)W^j(p)\Gamma^k_{ij}(F(p))V^lg_{kl}(F(p)) = V^iW^jZ(g_{ij}(F(p)))$$

$$Z(g_{ij}(F(p))) = Dg_{ij}(F(p))DF(p)[Z(p)] = g_{rj}\Gamma^r_{il}DF^l[Z(p)] + g_{ir}\Gamma^r_{lj}DF^l[Z(p)]$$

Up to relabelling, this concludes the proof of the claim.