3. Affine connections and geodesics

3.1. Connection along integral curves of vector fields.

Let X and Y be smooth vector fields on M, i.e. $X, Y \in \Gamma(TM)$. Let ∇ denote the Levi-Civita connection of (M, g). Let $p \in M$ and $\gamma : [0,1] \to M$ the integral curve of X through p. Recall that by definition, the integral curve γ satisfies the initial value problem

$$\begin{cases} \gamma(0) = p \\ \frac{\partial \gamma}{\partial t} = X(\gamma(t)) \,. \end{cases}$$

Prove that ∇ is given by

$$\nabla_X Y(p) = \frac{d}{dt}(P^{-1}_{\gamma,0,t}(Y(\gamma(t)))\Big|_{t=0} \,,$$

where $P_{\gamma,0,t} : T_{\gamma(0)}M \to T_{\gamma(t)}M$ is the parallel transport along γ from 0 to t.
Hint: prove that you can extend a basis $e_1, ..., e_m$ of T_pM to a local frame $e_1(t), ..., e_m(t) \in T_{\gamma(t)}M$ along γ such that $e_i(t)$ is parallel. Consider the restriction of Y on $\gamma([0,1])$ expressed using the parallel frame.

3.2. Connection is determined by geodesics and torsion tensor.

Let ∇ and $\tilde{\nabla}$ be two connections on TM. We call a curve $\gamma : [0,1] \to M$ a geodesic if it is self-parallel, that is $\nabla_{\partial_t} \gamma' = 0$ (see Definition 3.18 in the lecture notes). Prove that the following two are equivalent:

1. $\nabla \equiv \tilde{\nabla}$, the two connections are identical
2. ∇ and $\tilde{\nabla}$ have the same geodesics and the same torsion fields $T_\nabla \equiv T_{\tilde{\nabla}}$.

Hint: by Definition 3.5, $T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ and $T_\nabla(X,Y) = -T_\nabla(Y,X)$. By exercise 2.5, the difference of two connections satisfies $A = \nabla - \tilde{\nabla} \in \Gamma(T_{1,2}M)$. Decompose A in its symmetric A^s and anti-symmetric A^a part and try to understand what the conditions in 2. mean for A^a and A^s respectively.

3.3. Parallel transport on the 2-sphere.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere endowed with the Levi-Civita connection on TS^2. Let $\gamma : [0,1] \to S^2$ be a smooth curve.

1. Prove that parallel transport is independent of reparametrization: if V is a parallel vector field along γ, then for any diffeomorphism between intervals (reparametrization) $\alpha : [a,b] \to [0,1]$, $V \circ \alpha$ is a vector field along $\gamma \circ \alpha$ and it is parallel.
   Remark: For this part of the exercise, $M = S^2$ is not needed, and does not simplify the proof, the connection can be chosen to be arbitrary.

2. Prove that a vector field V along γ is parallel if and only if $\partial_t V(t) \in \mathbb{R}^3$ is normal to the sphere.
   Hint: use a parallel frame along γ (see also the hint to Problem 3.1) and the compatibility of the metric applied to sections of constant length.

3. What is the parallel transport along geodesics on S^2?
   Hint: By symmetry, it suffices to choose a geodesic in a convenient system of coordinates.

4. What is the parallel transport along curves that parametrize $S^2 \cap \{z = c\}$ for $c \in (-1,1)$?

3.4. Parallel transport and geodesics on hyperbolic plane.
Recall the hyperbolic plane introduced in exercise 1.4:

$$\mathbb{H}^2 := \{z \in \mathbb{C} \colon \Im z > 0\} \quad \text{and} \quad g_{x+iy} = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2).$$

1. Compute the Christoffel symbols of the Levi-Civita connection of $(\mathbb{H}^2, g)$.

2. Let $v = (0,1)$ be a tangent vector at $p = (0,1) \in \mathbb{H}^2$. (This means that under the canonical identification of $T_p\mathbb{H}^2$ with $\mathbb{R}^2$, the vector v_0 is identified with $(0,1)$.) Let $v(t)$ be the parallel transport of v along the curve $\gamma(t) = (t,1)$. Show that $v(t)$ makes an angle t with the direction of the y-axis.

3. Show that vertical lines are geodesics in $(\mathbb{H}^2, g)$. Show that any geodesic is either a vertical half line or a half circle intersecting the x-axis orthogonally.

Hint for 2: In the proof of 3.14 you saw the defining ODE for parallel transport in local coordinates. In this case, writing $v(t) = (w(t), z(t))$, deduce that the parallel transport system simplifies to

$$\begin{cases} \dot{w}(t) + \Gamma^1_{12}(\pi(v(t)))z = 0 \\ \dot{z}(t) + \Gamma^2_{11}(\pi(v(t)))w = 0 \end{cases} \tag{1}$$

Since parallel transport is an isometry, parametrize the curve $v(t)$ in $T\mathbb{H}^2$ using just one parameter $\theta(t)$, $v(t) = (\cos(\theta(t)), \sin(\theta(t)))$.

3. Solutions

Solution of 3.1: Recall the definition of parallel transport: defining for any initial value $e_0 \in T_{\gamma(0)}M$

$$e(t) = P_{\gamma,0,t}(e_0),$$

$$\nabla_{\dot\gamma} e = 0$$

Moreover, $P_{\gamma,0,t}$ is an isometry between $P_{\gamma(0)}$ and $P_{\gamma(t)}$. In particular, choosing one orthonormal basis $e_1, ..e_m$ of $T_{\gamma(0)M}$, one can use the parallel transport to obtain

$$\{P_{\gamma,0,t}(e_i) \,|\, i = 1, ..., m\}$$

which is a orthonormal basis for $T_{\gamma(t)}M$ given that P is an preserves scalar products. Using this orthonormal frame, we write for every point on the curve γ,

$$Y(\gamma(t)) = Y^i(t)e_i(t).$$

Using this notation we have

$$P^{-1}_{\gamma,0,t}(Y^i(t)e_i(t)) = Y^i(t)e_i(0)$$

and hence

$$\frac{d}{dt}(P^{-1}_{\gamma,0,t}(Y(\gamma(t)))\bigg|_{t=0} = \dot Y^i(0)e_i(0).$$

On the other hand, $\nabla_X Y \in \Gamma(TM)$ can be computed using the metric compatibility of the Levi-Civita connection:

$$g(\nabla_X Y, e_i) = X(Y^i) - g(Y, \nabla_X e^i) = X(Y^i) = \dot Y^i,$$

$$\implies \nabla_X Y = \dot Y^i(0)e_i\,.$$

Solution of 3.2: We prove 2. $\implies$ 1. (The other direction is clear)

$$A(X,Y) = \nabla_X Y - \tilde\nabla_X Y = A^a(X,Y) + A^s(X,Y)$$

for
$$A^a(X,Y) = \frac{1}{2}(A(X,Y) - A(Y,X)) \quad A^s(X,Y) = \frac{1}{2}(A(X,Y) + A(Y,X))$$
Then, $A^s(X,Y) = A^s(Y,X)$, $A^a(X,Y) = -A^a(Y,X)$ by definition.

$$A^s(X,X) = 0 \iff \nabla_X X = \tilde{\nabla}_X X.$$

$$A^a \equiv 0 \iff \nabla_X Y - \nabla_Y X = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X \equiv T_\nabla(X,Y) = T_{\tilde{\nabla}}(X,Y)$$

It remains to prove that $\nabla_X X \equiv \tilde{\nabla}_X X$ holds if and only if ∇ and $\tilde{\nabla}$ have the same geodesics. For fixed tangent vector $X(p) \in T_pM$ there exists a smooth curve $\gamma : [-I, I] \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X(p)$. If γ is a geodesic for a given connection ∇,

$$\nabla_X X = 0.$$

In particular if two connections have the same geodesics, $A^s \equiv 0$. Conversely, assume the two connections ∇ and $\tilde{\nabla}$ do not have the same geodesics. Then there is a point p and a curve γ, $\gamma'(0) = X$ such that γ is not a geodesic for $\tilde{\nabla}$ but it is for ∇. This implies the existence of t with
$$\nabla_X X(\gamma(t)) = 0 \quad \neq \quad \tilde{\nabla}_X X(\gamma(t)).$$

Solution of 3.3:

1. Denote by ∂_s the tangent vector on $[a,b]$ and by $\tilde{V}(s) := V(\gamma(\alpha(s)))$ the vector field along α. For a frame of TM along γ we write in local coordinates

$$V(\gamma(t)) = V^i(t)e_i(t) \quad \tilde{V}(s) = V^i(\alpha(s))e_i(\alpha(s))$$

$$\begin{aligned}\nabla_{\partial_s}\tilde{V} &= (\partial_s(V^k(\alpha(s))) + d(\gamma\circ\alpha)^i(\partial_s)V^j(\alpha(s))\Gamma^k_{ij})e_k \\ &= \partial_s\alpha(s)(\partial_t(V^k(((\alpha(s))))) + d(\gamma)^i(\partial_t)V^j(\alpha(s))\Gamma^k_{ij})e_k = \partial_s\alpha(s)\nabla_{\partial_t}V = 0\,.\end{aligned}$$

2. Let $(e_1(t), e_2(t))$ be a parallel orthonormal frame along γ. Let $V = V^i(t)e_i(t)$ be a vector field along γ.

$$\langle \nabla^{S^2}_{\partial_t} V, e_i\rangle = \langle \nabla^{\mathbb{R}^3}_{\partial_t} V, e_i\rangle = \langle \partial_t V, e_i\rangle = \nabla_{\partial_t} g(V_{ei})$$

and writing $V = V^i e_i$,
$$\partial_t V \perp S^2 \iff V^i \equiv \text{const.}$$

To conclude notice that

$$\partial_t g(V, e_i) = \partial_t V^i = g(\nabla_{\partial_t} V, e_i)$$

which implies that the coefficients V^i are constant if V is parallel. Moreover, linear combinations of parallel sections are parallel so that

$$\partial_t V \perp S^2 \iff V^i \equiv \text{const.} \iff \nabla^{S^2}_{\partial_t} V = 0.$$

3. Given 2., and assuming w.l.o.g. that we are considering the equatorial $S^1 \subset S^2$ as the geodesic γ, for any vector $v \in T_{(0,1,0)} S^2$, the parallel transport along γ is exactly the image of v under the rotation of $\mathbb{R}^3$ which fixes the z-axis.

4. Using polar coordinates $(\varphi, \theta) \in (0, 2\pi) \times (0, \pi)$ on S^2, such that φ is the coordinate describing the change in the planes parallel to $\{z = 0\}$, one can check that this frame is parallel when restricted to any curve parametrizing $\{z = 0\} \cap S^2$. Therefore in this case ($c = 0$) the parallel transport of a vector has constant coefficients in this frame.
When $c \neq 0$, the frame is no longer parallel. Let us first do some preliminary computations to recall the properties of these vector fields:

$$\partial_\theta (r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta)) = r(\cos(\theta) \cos(\varphi), \cos(\theta) \sin(\varphi), -\sin(\theta))$$

$$\partial_\theta(p) \cdot_{\mathbb{R}^3} p = r^2 \sin(\theta) \cos(\theta)(\sin^2(\varphi) + \cos^2(\varphi) - 1)) = 0$$

$$\partial_\varphi (r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta)) = r(-\sin(\theta) \sin(\varphi), \sin(\theta) \cos(\varphi), 0)$$

$$\partial_\varphi(p) \cdot_{\mathbb{R}^3} p = 0\,.$$

A parametrization of $S^2 \cap \{z = c\}$ is given by

$$\gamma : t \mapsto (\sin(\theta_c) \cos(t), \sin(\theta_c) \sin(t), \cos(\theta_c)).$$

The metric coefficients for the standard metric on S^2 in this frame are

$$g_{\theta\varphi} = 0 \quad g_{\varphi\varphi} = \sin^2(\theta) \quad g_{\theta\theta} = 1.$$

The Christoffel symbols are given by

$$\Gamma^\theta_{\varphi\varphi} = -\sin(\theta)\cos(\theta) \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{\cos(\theta)}{\sin(\theta)}$$

The parallel transport equation becomes

$$\nabla_{\partial_t} V = (\partial_t V^k + \partial_t \gamma^i V^j \Gamma^k_{ij})\partial_k = 0$$

$$\begin{cases} \partial_t V^\theta + \partial_t \gamma^\varphi V^\varphi \Gamma^\theta_{\varphi\varphi} = \partial_t V^\theta + \partial_t \gamma^\varphi V^\varphi (-\sin(\theta)\cos(\theta)) = 0 \\ \partial_t V^\varphi + \partial_t \gamma^\varphi V^\theta \Gamma^\varphi_{\varphi\theta} = \partial_t V^\varphi + \partial_t \gamma^\varphi V^\theta \frac{\cos(\theta)}{\sin(\theta)} = 0 \end{cases}$$

where we used $\gamma^\theta = \text{const.}$ to simplify the second equation. Using $\partial_t \gamma^\varphi = 1$,

$$\begin{cases} \partial_t V^\theta + V^\varphi (-\sin(\theta)\cos(\theta)) = 0 \\ \partial_t V^\varphi + V^\theta \frac{\cos(\theta)}{\sin(\theta)} = 0 \end{cases}$$

differentiating once more in t,

$$\begin{cases} \partial_t^2 V^\theta + \cos^2(\theta_c) V^\theta = 0 \\ \partial_t^2 V^\varphi + \cos^2(\theta_c) V^\varphi = 0 \end{cases}$$

$$\begin{cases} V^\theta = A\cos(\cos(\theta_c)t) + B\sin(\cos(\theta_c)t) \\ V^\varphi = C\cos(\cos(\theta_c)t) + D\sin(\cos(\theta_c)t)\,. \end{cases}$$

Solution of 3.4:

1. The metric coefficients with respect to the coordinates $\varphi = (\varphi^1, \varphi^2) = (x, y)$ are given by

   $$g_{12} = 0, g_{11} = g_{22} = \frac{1}{y^2}.$$

   The Christoffel symbols can then be computed using

   $$\Gamma^i_{kl} = \frac{1}{2} g^{im}(g_{mk,l} + g_{ml,k} - g_{kl,m})$$

   which gives

   $$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0 \quad \Gamma^2_{11} = \frac{1}{y} \quad \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$$

2. Following the hint we obtain that the parallel transport equation reduces to

   $$\frac{d\theta}{dt} = -1$$

   which together with $v_0 = (0, 1)$ gives $\theta(t) = \frac{\pi}{2} - t$.

3. If we use the Christoffel symbols obtained in 1. to simplify the geodesic equation we obtain the following characterizations of geodesics $t \mapsto (x(t), y(t))$:

$$\frac{d^2x}{dt^2} - \frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} = 0 \quad \frac{d^2y}{dt^2} + \frac{1}{y}\left(\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right) = 0$$

If $\frac{dx}{dt} \neq 0$,

$$\frac{d}{dt}\left(y\frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1}\right) = \left(\frac{dy}{dt}\right)^2\left(\frac{dx}{dt}\right)^{-1} + y\frac{d^2y}{dt^2}\left(\frac{dx}{dt}\right)^{-1} - y\frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-2}\frac{d^2x}{dt^2}$$

and inserting the geodesic equations from above,

$$\frac{d}{dt}\left(y\frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1}\right) = y\frac{d^2y}{dt^2}\left(\frac{dx}{dt}\right)^{-1} - \left(\frac{dy}{dt}\right)^2\left(\frac{dx}{dt}\right)^{-1}$$

$$= -\left(\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right)\left(\frac{dx}{dt}\right)^{-1} - \left(\frac{dy}{dt}\right)^2\left(\frac{dx}{dt}\right)^{-1} = -\frac{dx}{dt}$$

By rearranging terms we find the equation

$$\frac{d^2}{dt}(x^2 + y^2) = 0$$

which is the equation of circles centered at the origin. The case $\frac{dx}{dt}$ corresponds to vertical straight lines.