4. Geodesics, Hopf-Rinow theorem

4.1. Geodesic variations.

1. Let $\gamma : [0,1] \to M$ be a smooth curve in a Riemannian manifold (M,g). Let V be a vector field along γ with $V(0) = V(1) = 0$. Show that there exists
$$\tilde{\gamma} : (-1,1) \times [0,1] \to M$$
satisfying $\tilde{\gamma}(0,t) = \gamma(t)$, $\tilde{\gamma}(0,s) = \gamma(0)$, $\tilde{\gamma}(1,s) = \gamma(1)$ and such that V is the variation vector field of $\tilde{\gamma}$, i.e. $V(t) = \partial_s \tilde{\gamma}(s,t)$.

2. Let $\gamma : [0,1] \times [0,a] \to M$ be a smooth map such that for fixed $a_0 \in [0,a]$, $\gamma_{a_0} : t \mapsto \gamma(t,a_0)$ is a geodesic parametrized by arc length. Assume that the curve $b \mapsto \gamma(0,b)$ is orthogonal to the curve γ_{a_0} at the point $\gamma(0,a_0)$. Prove that for all $(t_0,a_0) \in [0,1] \times [0,a]$ the curves $b \mapsto \gamma(t_0,b)$ and γ_{a_0} are orthogonal where they intersect.

4.2. Exponential map on $SO(n)$.

Consider $M = SO(n) \subset \mathbb{R}^{n\times n}$ with the induced metric. Consider $p = I \in SO(n)$, show that

1. $T_pM = \{B \in \mathbb{R}^{n\times n} \mid B + B^T = 0\}$
2. $\exp_p(B) = \sum\limits_{i=0}^{\infty} \frac{1}{i!} B^i$ (matrix exponential).

4.3. Riemannian structure on TTM.

Let $(p,v) \in TM$ and $V, W \in T_{(p,v)}TM$. Choose curves in TM with
$$\alpha : t \mapsto (p(t), v(t)) \quad \beta : s \mapsto (q(s), w(s)),$$
$$p(0) = q(0) = p \quad v(0) = w(0) = v \quad \alpha'(0) = V \quad \beta'(0) = W.$$
Define an inner product on TM by
$$g_{(p,v)}(V,W) = g_p^M(d\pi(V), d\pi(W)) + g_p^M(\nabla^M_{\partial_t} v(0), \nabla^M_{\partial_t} w(0)) \,.$$

1. Prove that this formula defines a well-defined Riemannian metric on TM.

2. A vector $(p, v) \in TM$ that is orthogonal to the vectors tangent to the fiber $\pi^{-1}(p) \cong T_pM$ is called a hotizontal vector. A curve $\gamma : t \mapsto (p(t), v(t)) \in TM$ is defined to be horizontal if its tangent vector $\gamma'(t) \in T_{\gamma(t)}TM$ is horizontal for all t.
   Prove that the curve $\gamma(t)$ is horizontal if and only if $v(t)$ is parallel along $p(t)$ with respect to the Riemannian structure and Levi-Civita connection of M.

3. Prove that the geodesic vector field on TTM (see also proof of 4.1) is horizontal at every point.

4. Prove that the flow lines of the geodesic vector field are geodesics on TM for the metric introduced in 1.

4.4. Applications of Hopf-Rinow.

Let (M, g) be a homogeneous Riemannian manifold, i.e. the isometry group of M acts transitively on M. Prove that for any two points $p, q \in M$ there exists a geodesic γ between them satisfying $L(\gamma) = d(p, q)$.

4.5. Existence of closed geodesics.

Let (M, g) be a compact Riemannian manifold and $c_0 \colon S^1 \to M$ a continuous closed curve. The purpose of this exercise is to show that in the family of all continuous and piece-wise C^1 curves $c \colon S^1 \to M$ which are homotopic to c_0, there is a shortest one and it is a geodesic.

a) Show that c_0 is homotopic to a piece-wise C^1-curve c_1 with finite length.

b) Let $L \coloneqq \inf_c L(c)$ be the infimum over all piece-wise C^1 curves $c \colon S^1 \to M$ homotopic to c_0 and consider a minimizing sequence $(c_n \colon S^1 \to M)_n$ with $\lim_n L(c_n) = L$. Use compactness of M to construct a piece-wise C^1-curve $c \colon S^1 \to M$ with length L.
   Hint. Cover M with simply connected balls with the property that every two points in a ball are joined by a unique distance minimizing geodesic.

c) Conclude by showing that c is homotopic to c_0 and a geodesic.

4. Solutions

Solution of 4.1:

1. Since our curve is embedded, V can be extended to a smooth compactly supported vector field $\tilde{V} \in \Gamma(TM)$ which agrees with V on γ. Since the support of $\tilde{V}$ is compact, its flow ϕ_t is defined for all times. Define

$$\tilde{\gamma}(s,t) = \psi_s(\gamma(t)).$$

Then $\tilde{\gamma}(0,t) = \psi_0(\gamma(t)) = id(\gamma(t)) = \gamma(t)$. Moreover $\tilde{\gamma}(s,0) = \psi_s(\gamma(0)) = \gamma(0)$ since $V(\gamma(0)) = 0$ implies that the point is a fixed point for the flow. The same holds for $\gamma(1)$.

2. $$\frac{d}{dt} g(\partial_t\gamma, \partial_s\gamma) = g(\nabla_t\partial_t\gamma, \partial_s\gamma) + g(\partial_t\gamma, \nabla_t\partial_s\gamma) = \frac{1}{2}\frac{d}{ds} g(\partial_t\gamma, \partial_t\gamma) = 0$$

From this computation we deduce that the two curves are orthogonal where they intersect, as long as this holds at one point.

Solution of 4.2: We will present a proof using notation that will be introduced this week.

Solution of 4.3:

1. g is symmetric and bilinear by definition. It remains to check that it is positive definite.
Let $\{x_1, ..., x_m\}$ be coordinates for M around p and let $\{x_1, ..., x_m, y_1, ..., y_m\}$ be the coordinates on TM obtained from the first chart ($v = \sum_{i=1}^m y_i \frac{\partial}{\partial x_i}$). The tangent vectors $V \in T_{p,v}TM$ can be represented in local coordinates

$$V = \sum_{i=1}^m \alpha^i \frac{\partial}{\partial x_i} + \theta^i \frac{\partial}{\partial y_i}$$

Assume $g_{p,v}(V,V) = 0$, then both terms have to vanish:

$$g_p^M(d\pi(V), d\pi(V)) = 0 \quad g_p^M(\nabla^M_{\partial_t} v(0), \nabla^M_{\partial_t} v(0)) = 0$$

and since g^M is positive definite this implies

$$d\pi(V) = 0 \quad \nabla_{\partial_t} v(0) = 0\,.$$

Using the local coordinates from above,

$$d\pi_{p,v}(V) = \sum_{i=1}^{m} \alpha^i \frac{\partial}{\partial x_i}$$

$$(\nabla_{\partial_t} v(0))^k = \dot{v}^k(0) + \Gamma^k_{ij} dp^i(\partial_t) v^j(0) = \theta^k + \Gamma^k_{ij} \alpha^i v^j(0)\,.$$

Positive definiteness now follows from the fact that $d\pi(V) = 0 \implies \alpha^i = 0$ for all i and given $\alpha^i \equiv 0$, $\nabla_{\partial_t} v(0) = 0 \implies \theta^k = 0$ for all k.

2. Using the notation from part 1, a vector V is horizontal if the following holds: for all $W = (\dot{q}(0), \dot{\beta}(0)) \in T_{p,v}TM$ such that $d\pi_{p,v}(W) = 0$,

$$g_{TM}(V, W) = 0 = g^M(\nabla_{\partial_t} v(0), \nabla_{\partial_t} \beta(0))$$

Choosing the vector $W = (\dot{q}(0), \dot{v}(0))$ we see that $\nabla_{\partial_t} v(0) = 0$. This can be done at any point of the curve and we conclude that the curve is horizontal if and only if it is parallel.

3+4. Let $\gamma(t) = (p(t), v(t))$ be a curve in TTM. Then, by definition of the metric g_{TM},

$$L^M(p(t)) \le L^{TM}(\gamma(t)) \text{ with equality iff } \gamma \text{ is horizontal}$$

By the local minimizing properties of geodesics, $\gamma(t) = (c(t), c'(t))$ is a geodesic in TM whenever $c(t)$ is a geodesic in M. Since geodesics are the flow lines of the geodesic vector field the claims follows.

Solution of 4.4: Let $p \in M$. Pick $r > 0$ such that $\exp_p$ is defined on $B(0, r) \subset TM_p$. Let $v \in T_pM$ be a tangent vector and let (α_v, ω_v) be the maximal interval, where the geodesic c_v satisfying $c_v(0) = p$ and $\dot{c}_v(0) = v$ is defined. We need to show that $(\alpha_v, \omega_v) = (-\infty, \infty)$. Suppose that $\omega_v < \infty$. Let $0 < \epsilon < r$. Consider $q = c_v(\omega_v - \epsilon) \in M$. By assumption, there exists an isometry Φ of M such that $\Phi(p) = q$. Put $w := D\Phi_q^{-1}(\dot{c}_v(\omega_v - \epsilon)) \in T_pM$ and let c_w be the associated geodesic. Then $\Phi \circ c_w$ is a geodesic starting at q that extends c_v to $(\alpha_v, \omega_v + r - \epsilon)$. This is a contradiction to the maximality of ω_v. Hence $\omega_v = \infty$. Similarly one shows $\alpha_v = -\infty$.

This shows that $\exp_p(tv)$ is defined on $(-\infty, \infty)$ and therefore we can prove the claim using the theorem of Hopf-Rinow.

Solution of 4.5: a) Let us first prove that c_0 is homotopic to a piece-wise C^1-curve c_1. To this aim, we split c_0 into finitely many paths $\gamma_i\colon [0,1] \to M$ such that $\gamma_i(1) = \gamma_{i+1}(0)$, $\gamma_n(1) = \gamma_1(0)$ and γ_i is contained in a charts $\{(\phi_i, U_i)\}_{i=1}^n$ with U_i simply-connected. Then γ_i is homotopic (relative to the endpoints) to a C^1-curve $\tilde{\gamma}_i$ and by connecting the $\tilde{\gamma}_i$'s we get a piece-wise C^1-curve c_1 which is homotopic to c_0. Then c_1 has finite length $L(c_1)$.

b) Let $L := \inf_c L(c) < \infty$ be the infimum over all curves $c\colon S^1 \to M$ which are piece-wise C^1 and homotopic to c_0 and consider a minimizing sequence, i.e. a sequence $(c_n\colon S^1 \to M)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} L(c_n) = L$.

We may assume that the curves $c_n\colon [0,1] \to M$ are parametrized proportional to arclength, i.e. $L(c_n|_{[a,b]}) = |b-a| \cdot L(c_n)$.

As M is compact, there is some $r > 0$ and points $q_q, \dots, q_n \in M$ such that the balls $B(q_1, r), \dots, B(q_n, r)$ cover M, for all $q, q' \in B(q_i, 3r)$ there is a unique distance minimizing geodesic joining q to q' of length $< 6r$ and the balls $B(q_i, 6r)$ are simply connected.

Fix some $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{r}{L}$ and define $t_k := \frac{k}{N}$ for $k = 0, \dots, N$. Consider now the sequences $(c_n(t_k))_{n\in\mathbb{N}}$. By compactness of M, we may assume (by possibly passing to subsequences) that $c_n(t_k) \to p_k$ for each $k = 0, \dots, N$. Therefore

$$d(p_k, p_{k+1}) \leq \limsup_{n\to\infty} d(c_n(t_k), c_n(t_{k+1})) \leq \limsup_{n\to\infty} \tfrac{1}{N} L(c_n) < r.$$

Take $\left\{ {}^{q\in}_{q_1,\dots,q_n} \right\}$ such that $p_k \in B(q, r)$, then $p_{k+1} \subset B(q, 3r)$ and therefore we can define a continuous, piece-wise C^1-curve $c\colon [0,1] \to M$ by concatenating the unique distance minimizing geodesics between p_k and p_{k+1}.

For the length of c we have

$$L(c) = \sum_{k=0}^{N-1} L\left(c|_{[t_k, t_{k+1}]}\right) = \sum_{k=0}^{N-1} d(p_k, p_{k+1}) \leq N \limsup_{n\to\infty} \tfrac{1}{N} L(c_n) = L.$$

c) It remains to prove that c is homotopic to c_0. Observe that for n large enough, we have $c([t_k, t_{k+1}]), c_n([t_k, t_{k+1}]) \subset B(q, 3r)$.

Since $B(q, 6r)$ is simply-connected there is a homotopy from $c_n|_{\left[\frac{k}{N}, \frac{k+1}{N}\right]}$ to $c|_{\left[\frac{k}{N}, \frac{k+1}{N}\right]}$ with the endpoints following the unique geodesics from $c_n(t_k)$ to p_k and from $c_n(t_{k+1})$ to p_{k+1}, respectively. Combining this homotopies, we get a homotopy from c_n to c.

Observe that c is locally length minimizing and hence is a geodesic.