5. Curvature

5.1. Ricci curvature.

Let (M, g) be a 3-dimensional Riemannian manifold. Show the following:

1. The Ricci curvature Ric uniquely determines the Riemannian curvature tensor R.
2. If M is an Einstein manifold, that is, a Riemannian manifold (M, g) with $\mathrm{Ric} = kg$ for some $k \in \mathbb{R}$, then the sectional curvature sec is constant.

5.2. Metric and Riemannian isometries.

Let (M, g) and $(\bar{M}, \bar{g})$ be two connected Riemannian manifolds with induced distance functions d and $\bar{d}$, respectively. Further, let $f\colon (M, d) \to (\bar{M}, \bar{d})$ be an isometry of metric spaces, i.e. f is surjective and for all $p, p' \in M$ we have $\bar{d}(f(p), f(p')) = d(p, p')$.

1. Prove that for every geodesic γ in M, $\bar{\gamma} := f \circ \gamma$ is a geodesic in N.
2. Let $p \in M$. Define $F\colon TM_p \to T\bar{M}_{f(p)}$ with
$$F(X) := \frac{d}{dt}\bigg|_{t=0} f \circ \gamma_X(t),$$
where γ_X is the geodesic with $\gamma_X(0) = p$ and $\dot{\gamma}(0) = X$. Show that F is surjective and satisfies $F(cX) = cF(X)$ for all $X \in TM_p$ and $c \in \mathbb{R}$.
3. Conclude that F is an isometry by proving $\|F(X)\| = \|X\|$.
4. Prove that F is linear and conclude that f is smooth in a neighborhood of p.
5. Prove that f is a diffeomorphism for which $f^*\bar{g} = g$ holds.

5.3. Flat manifolds.

Consider the torus $\mathbb{T}^m = S^1 \times ... \times S^1$ endowed with the product metric coming from m-times the standard metric on S^1.

1. Express the metric g in local coordinates.
2. Show that this metric on the torus $(\mathbb{T}^m, g)$ defines a flat manifold (a manifold for which $K(\Pi) = 0$ for every plane $\Pi \subset T_pM$ and every $p \in M$).
3. Decide whether this statement is true or false: "A smooth Riemannian manifold is flat if and only if the Riemann curvature tensor vanishes identically."

5.4. Curvatures of spheres.

Let $S_r^m \subset \mathbb{R}^{m+1}$ be the m-dimensional sphere of radius r endowed with the standard metric.

1. Compute the Riemann curvature tensor of S_r^m.
2. Compute the Ricci curvature tensor of S_r^m.
3. Compute the scalar curvature of S_r^m.

5. Solutions

Solution of 5.1:

1. In the following, let e_1, e_2, e_3 be an orthonormal basis of TM_p. First, note that $R_{iijk} = R_{jkii} = 0$ by the symmetry properties of R.

   We denote the components of Ric by R_{ij}. Then, for $\{i,j,k\} = \{1,2,3\}$, we have

   $$\begin{aligned} R_{ii} &= R_{iiii} + R_{jiji} + R_{kiki} = R_{ijij} + R_{ikik}, \\ R_{ij} &= R_{iiij} + R_{jijj} + R_{kikj} = R_{ikjk} \end{aligned}$$

   and therefore, we get

   $$\begin{aligned} 2R_{ijij} &= R_{ii} + R_{jj} - R_{kk}, \\ R_{ikjk} &= R_{ij}. \end{aligned}$$

   Observe now, that we can compute all other components of R by symmetry properties. Hence R is uniquely determined by Ric.

2. Let e_1, e_2 be a orthonormal basis of $E \subset TM_p$ and choose e_3 such that e_1, e_2, e_3 is an orthonormal basis of TM_p. Then we have

   $$2\sec_p(E) = 2R_{1212} = R_{11} + R_{22} - R_{33} = k + k - k = k$$

   and hence $\sec_p(E) = \frac{k}{2}$.

Solution of 5.2:

1. As the property of being a geodesic is local, we may assume that both $\gamma\colon [0,L] \to M$ and $f \circ \gamma\colon [0,L] \to \bar{M}$ are contained in an open set $U \subset M$ and $\bar{U} \subset \bar{M}$, respectively, such that points in U and $\bar{U}$ are connected by a unique distance minimizing geodesic. Then there is a unique geodesic β from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$. We claim that $\bar{\gamma}$ and β coincide.

   In the following all geodesics are parametrized by arclength. For $t \in [0,L]$ there are geodesics β_1 from $\bar{\gamma}(0)$ to $\bar{\gamma}(t)$ and β_2 from $\bar{\gamma}(t)$ to $\bar{\gamma}(L)$. Concatenating β_1 and β_2, we get some piece-wise C^1-curve from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$ with length

$$\begin{aligned} L(\beta_1\beta_2) &= L(\beta_1) + L(\beta_2) \\ &= \bar{d}(\bar{\gamma}(0), \bar{\gamma}(t)) + \bar{d}(\bar{\gamma}(t), \bar{\gamma}(L)) \\ &= d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(L)) \\ &= d(\gamma(0), \gamma(L)) = \bar{d}(\bar{\gamma}(0), \bar{\gamma}(L)) = L(\beta). \end{aligned}$$

   Hence, by uniqueness of the geodesic from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$, $\beta_1\beta_2$ and β coincide, i.e. $\bar{\gamma}(t) = \beta(t)$.

2. Observe that f is bijective and its inverse f^{-1} is also is an isometry of metric spaces.

   First, we prove that F is surjective. Let $Y \in T\bar{M}_{f(p)}$ and $\bar{\gamma}$ the geodesic through $f(p)$ with $\dot{\bar{\gamma}}(0) = Y$. Then $Y = F(X)$ for $X := \frac{d}{dt}\big|_{t=0} f^{-1} \circ \bar{\gamma}(t)$.

   From $\gamma_{cX}(t) = \gamma_X(ct)$ it follows that

$$F(cX) = \frac{d}{dt}\bigg|_{t=0} f \circ \gamma_X(ct) = cF(X).$$

3. For $\epsilon > 0$ small enough, we have that $\gamma_X(\epsilon)$ and $f \circ \gamma_X(\epsilon)$ are contained in a normal neighborhood of p and $f(p)$, respectively. Hence we get

$$\epsilon\|X\| = d(p, \gamma_X(\epsilon)) = \bar{d}(f(p), f \circ \gamma_X(\epsilon)) = \epsilon\|F(X)\|.$$

   We now claim that for $X, Y \in TM_p$ with $\|X\| = \|Y\| = 1$ and α such that $\cos\alpha = g_p(X,Y)$ we have

$$\sin\frac{1}{2}\alpha = \lim_{s\to 0} \frac{1}{2s} d(\gamma_X(s), \gamma_Y(s)),$$

   and a similar formula for $\bar{X}, \bar{Y} \in T\bar{M}_f(p)$ with $\|\bar{X}\| = \|\bar{Y}\| = 1$.

Assuming the claim for the moment, we now prove that

$$g_p(X,Y) = \bar{g}_{f(p)}(F(X),F(Y)).$$

for all $X, Y \in TM_p$.

Note first that since $F(cX) = cF(X)$, we can assume that $\|X\| = \|Y\| = 1$, then also $\|F(X)\| = \|F(Y)\| = 1$. So by the claim and the fact that f is a distance preserving map we have for $\cos\alpha = g_p(X,Y)$ and $\cos\alpha' = \bar{g}_{f(p)}(F(X),F(Y))$

$$\sin\frac{1}{2}\alpha = \sin\frac{1}{2}\alpha'.$$

Therefore

$$g_p(X,Y) = \cos\alpha = 1 - 2\sin^2\frac{1}{2}\alpha = 1 - 2\sin^2\frac{1}{2}\alpha' = \bar{g}_{f(p)}(F(X),F(Y)).$$

4. For all $X, Y, Z \in TM_p$ and $c \in \mathbb{R}$, we have

$$\begin{aligned}
\bar{g}_{f(p)}(F(X+cY),F(Z)) &= g_p(X+cY,Z) \\
&= g_p(X,Z) + cg_p(Y,Z) \\
&= \bar{g}_{f(p)}(F(X),F(Z)) + c\bar{g}_{f(p)}(F(Y),F(Z)) \\
&= \bar{g}_{f(p)}(F(X)+cF(Y),F(Z))
\end{aligned}$$

Hence F is linear and therefore smooth.

If V_p is a neighborhood of $0 \in TM_p$ such that $\exp_p|_{V_p}\colon V_p \to U_p$ is a diffeomorphism, then we have

$$f|_{U_p} = \exp_{f(p)} \circ F \circ (\exp_p|_{V_p})^{-1}.$$

Hence f is smooth as well.

5. The argument above works for all $p \in M$ and also for f^{-1}. Hence f is a diffeomorphism. Furthermore, we have

$$df_p = d(\exp_{f(p)} \circ F \circ \exp_p^{-1}) = F$$

and thus

$$f^*\bar{g}_p(X_p,Y_p) = \bar{g}_{f(p)}(df_p(X_p),df_p(Y_p)) = \bar{g}_{f(p)}(F(X_p),F(Y_p)) = g_p(X_p,Y_p),$$

for all $X, Y \in TM$.

Proof of the claim (sketch). Let $X, Y \in TM_p$ with $\|X\| = \|Y\| = 1$ and let $\alpha = \sphericalangle_0(X, Y)$, that is, $\cos\alpha = g_p(X, Y)$. Consider normal coordinates $(\varphi, B(p, r))$ around p, so that we have $\varphi\colon B(p, r) \to B_r \subset \mathbb{R}^n$ and define $c_X := \varphi \circ \gamma_X$ and $c_Y := \varphi \circ \gamma_Y$, two curves in B_r.

On B_r we can consider two different metrics. The Euclidean metric g_E and the pull-back metric $h := (\varphi^{-1})^* g$.

Note that $h_0(c'_X(0), c'_Y(0)) = g_p(X, Y)$ and by Lemma 1.19 $h_0 = (g_E)_0$, so $(g_E)_0(c'_X(0), c'_Y(0)) = g_p(X, Y)$. We are now in a completely Euclidean setting.

Suppose by contradiction that $\limsup_{s\to 0} \frac{1}{2s} d(\gamma_X(s), \gamma_Y(s)) > \sin\frac{1}{2}\alpha$ and take $c > 1$ such that

$$\limsup_{s\to 0} \tfrac{1}{2s} d(\gamma_X(s), \gamma_Y(s)) > c \sin\tfrac{1}{2}\alpha.$$

Now, take r small enough such that $c^{-1} \cdot g_E < h < c \cdot g_E$ on $B_r \subset \mathbb{R}^n$, and therefore

$$c^{-1} \cdot d_E < d_h < c \cdot d_E,$$

where d_h denotes the distance function induced by the metric h. This implies that for s small enough

$$\frac{1}{2s} d_E(c_X(s), c_Y(s)) > c \sin\frac{1}{2}\alpha,$$

which is not true. The other inequality follows similarly. $\square$

Solution of 5.3:

1. Denoting by θ_i the canonical coordinate on S^1_i, the metric on $\mathbb{T}^m$ can be written as

$$g = d\theta_1 \otimes d\theta_1 + ... + d\theta_m \otimes d\theta_m\,.$$

2. The metric coefficients g_{ij} in the frame above are constant, therefore by Lemma 3.8 the Christoffel symbols vanish, that is

$$\Gamma^i_{jk} = 0 \quad \text{for all } i, j, k\,.$$

By the formula for the Riemann tensor coefficients in local coordinates (see also remarks after Proposition 3.8) we conclude that $R \equiv 0$.

3. The statement is true. This follows immediately from Lemma 5.8.

Solution of 5.4:

1. By Proposition 5.9. if the sectional curvature is constant, then the Riemann curvature tensor is given by

$$R(X,Y,Z,W) = k_0\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle \,.$$

   By example 5.10 that the image under the exponential map exp_p of a two-plane $\Pi \subset T_pM$ is isometric to the two-sphere (of radius r) and the sectional curvature is therefore constant $k_0 = \frac{1}{r^2}$ (independent of the plane chosen) at any point.

2. $\mathrm{Ric} = (m-1)g$, see also 5.15.

3. By definition 5.20, $\mathrm{scal}_p = \frac{m(m-1)}{r^2}$.