6. Curvature of submanifolds, Lie groups

6.1. Liouville's theorem.

Let X be the geodesic vector field on TM.

1. Prove that $\operatorname{div}(X) = 0$.
2. Prove that the geodesic flow preserves the volume of TM.

6.2. Totally geodesic submanifolds.

Prove the following proposition.
For a submanifold $M \subset \tilde{M}$ the following four statements are equivalent:

1. vanishing second fundamental form, i.e. $\kappa(X,Y) = 0$ for all $X, Y \in TM$
2. every geodesic in M is also a geodesic in $\tilde{M}$
3. if $v \in T_pM$ then the unique geodesic $\tilde{\gamma}$ of $\tilde{M}$ with initial velocity v lies initially in M
4. if $c : I \to M$ is a curve, then every ∇-parallel vector field $Y \in \Gamma(c^*(TM))$ is also $\tilde{\nabla}$ parallel.

6.3. Pull-back connections and curvature.

1. Prove the following Proposition.
   Let R be the curvature tensor of a connection ∇ on TM. For another manifold N and a smooth map $F : N \to M$, let R^F denote the curvature tensor of the connection ∇^F along F, given by

   $$R^F(X,Y)W := \nabla^F_X \nabla^F_Y W - \nabla^F_Y \nabla^F_X W - \nabla^F_{[X,Y]} W \in \Gamma(F^*TM)$$

   for $X, Y \in \Gamma(TN)$ and $W \in \Gamma(F^*TM)$). Then

   $$(R^F(X,Y)W)_p = R(F_*X_p.F_*Y_p)W_p$$

   for all $p \in N$.
   Remark: the notation for the pull-back connection along a curve c used in the lectures is ∇_{∂_s} (∇^c in the notation of the proposition).

2. If $F\colon (-1,1)_s \times (0,1)_t \to M$ is a smooth map, then $[D/ds, D/dt]V = R(V, \gamma')V$ for $V(t) = \partial_s F(0,t)$, $\gamma'(t) = \partial_t F(0,t)$.

6.4. Lie groups.

Definition: A Lie group G is a smooth manifold G which has a smooth group structure (m and i are smooth maps between smooth manifolds)

$$m : G \times G \to G \quad m(p, q) = pq$$

$$i : G \to G \quad i(p) = p^{-1}\,.$$

For any $p \in G$, the left-multiplication $L_p : G \to G$ defined by $L_p(q) = pq$ is a diffeomorphism. A Riemannian metric g on G is called left-invariant if $g(v, w) = g(dL_p(v), dL_p(w))$ for all $v, w \in TG$ and $p \in G$. One defines right-multiplication and right invariance analogously. A metric is called bi-invariant if it is both right- and left-invariant.
The Lie algebra $T_eG =: \mathcal{G}$ can therefore be identified with the set of left-invariant vector fields on G.

1. Let $X \in \mathcal{G}$ be a left-invariant vector field. Prove that the flow ϕ_t of X is defined globally for all times t and that the flow lines $\gamma : \mathbb{R} \to G$ with $\gamma(0) = e$ satisfy $\gamma(t + s) = \gamma_t \cdot \gamma_s(a)$.
   Remark: a map $\gamma : \mathbb{R} \to G$ is called a 1-parameter subgroup of G if it satisfies $\gamma(t + s) = \gamma(t) \cdot \gamma(s)$.
2. Prove that if G has a bi-invariant metric g, then the geodesics starting at e are 1-parameter subgroups of G.

Let G be a Lie group with bi-invariant metric g. Let $X, Y, Z \in \Gamma(TG)$ be left-invariant vector fields.

3. Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.
   Hint: show that $\nabla_X X = 0$ for left invariant vector fields.
4. Prove using (3.) that $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.
5. Prove that if X and Y are orthonormal, the sectional curvature of the plane Π generated by X and Y is given by $K(\Pi) = \frac{1}{4}g([X, Y], [X, Y])$. Conclude that if the flows of X and Y commute, the planes Π have vanishing sectional curvature.
   Remark: this shows that the sectional curvature of a Lie group with bi-invariant metric is non-negative.

6. Solutions

Solution of 6.1: Let $p \in M$ and consider the system of normal coordinates at p given by considering an orthonormal basis $\{e_i\}_i$ of T_pM and defining the chart u as follows:

$$q = exp_p(\sum_i u_i e_i) \quad (u_1, ..., u_n) = u(q)\,.$$

In this coordinate system, the Christoffel symbols vanish at p,

$$\Gamma^i_{jk}(p) = 0$$

and hence for $X = x^i \partial u_i$

$$div(X)(p) = \sum_i \frac{\partial x^i}{\partial u_i}$$

Let (u_i, v_j), $v = \sum_j v_j \partial u_j$ be coordinates on TM at (q, v). In these coordinates (u_i, v_j)

$$G(u_i) = v_i \quad G(v_j) = \sum_{ik} \Gamma^j_{ik} v_i v_k$$

$$div(G) = 0\,.$$

Solution of 6.2: We first prove an intermediate result:
Lemma: For a curve $c : I \to M \subset \bar{M}$ and a vector field $Y \in \Gamma(c^*TM)$ along c,

$$\frac{\overline{D}}{dt} Y = \frac{D}{dt} Y + \kappa(\dot{c}, \dot{c}).$$

In particular c is a geodesic in M if and only if $\frac{\overline{D}}{dt}\dot{c}$ is normal to M.
Proof. Locally in a parallel frame $A_1, ..., A_n$ along c, writing $Y = \sum_i Y^i A_i \circ c$,

$$\frac{\overline{D}}{dt} Y - \frac{D}{dt} Y = \sum_i Y^i \left(\frac{\overline{D}}{dt}(A_i \circ c) - \frac{D}{dt}(A_i \circ c)\right) = \sum_i Y^i (\overline{\nabla}_{\dot{c}} A_i - \nabla_{\dot{c}} A_i) = \sum_i Y^i \kappa(\dot{c}, A_i \circ c)\,.$$

□

We now move on to the proof of the proposition:
Proof.
By the lemma above, 1. $\implies$ 4.
By taking $Y = \dot{c}$, 4. $\implies$ 2.
2. $\implies$ 3. is clear.
By the lemma above, 3. implies that $\kappa(v, v) = 0$ for all $v \in TM$ which in turn implies $\kappa \equiv 0$ since the second fundamental form is symmetric. □

Solution of 6.3:

1. Proof. : Let $A_1, ..., A_m$ be coordinate vector fields on an open set $U \subset M$. Suppose that X, Y are vector fields on $F^{-1}(U)$, and write $F_*Y = \sum_j Y^j A_j \circ F$ for smooth functions X^j and Y^j on $F^{-1}(U)$. Furthermore, suppose that $W = A \circ F$ on $F^{-1}(U)$ for a vector field A on U, and put $B_k := \nabla_{A_k} A$. Then $\nabla^F_Y(A \circ F) = \nabla_{F_*Y} A = \sum_j Y^j B_j \circ F$, hence

$$\nabla^F_X \nabla^F_Y (A \circ F) = \sum_j X(Y^j) B_j \circ F + \sum_j Y^j \nabla^F_X (B_j \circ F),$$

and the second sum equals

$$\sum_j Y^j \nabla_{F_*X} B_j = \sum_{ij} X^i Y^j (\nabla_{A_i} B_j) \circ F.$$

Since $[A_i, A_j] = 0$,

$$\nabla^F_{[X,Y]}(A \circ F) = \nabla \sum_j (X(Y^j) - Y(X^j) B_j \circ F.$$

Combining these identities yields

$$\begin{aligned} R^F(X,Y)(A \circ F) &= \sum_{i,j} X^i Y^j (\nabla_{A_i} B_j - \nabla_{A_j} B_i) \circ F = \sum_{ij} X^i Y^j R(A_i, A_j) A \circ F \\ &= R(F_*X, F_*Y)(A \circ F). \end{aligned}$$

Solution of 6.4:

1. Assume that the flow line has a maximal time of existence t_0 and $\gamma(t_0) = p_0$, $\gamma'(t_0) = X_0 = X(t_0)$, where X is a left invariant vector field. Then, define

$$\gamma(t) = \exp_{p_0}(X_0 t), t \in (-\epsilon, \epsilon)\,.$$

$$\frac{d}{dt}|_{t=t_1} \gamma(t) = \frac{d}{dt}|_{t=0} \gamma(t + t_1) = \frac{d}{dt}|_{t=0} (\exp(t_1 X_0 + t X_0))$$

$$= \frac{d}{dt}|_{t=0} (\exp(t_1 X_0) \exp(t X_0)) = \frac{d}{dt}|_{t=0} (\gamma(t_1) \exp(tX)) = T_e(l_{\gamma(t_1)})(X) = X(\gamma(t_1)),$$

where we used the left invariance of the vector field in the last step. This proves that the flow lines are defined for all times. To prove the second property, define

$$\delta(t) = \gamma(s)^{-1}\gamma(s+t)$$

and differentiate to obtain

$$\delta'(t) = Dl_{\gamma(s)^{-1}}(\gamma(s+t))[X(\gamma(s+t))] = X(\gamma(s)^{-1}\gamma(s+t)) = X(\delta(t))$$

where the third step follows by left-invariance. By uniqueness of integral curves,

$$\delta(t) = \gamma(t)$$

and the claim follows.

2. Assuming the hint below, $\nabla_X X = 0$ for left invariant vector fields. Hence, integral curves of left invariant vector fields are exactly the geodesics of G.

3. Since both X and Y are left invariant, so is $X+Y$. By the hint,

$$\nabla_{X+Y} X + Y = \nabla_X Y + \nabla_Y X = 0.$$

This implies

$$\nabla_X Y = \frac{1}{2}(\nabla_X Y - \nabla_Y X) + \frac{1}{2}(\nabla_X Y + \nabla_Y X) = \frac{1}{2}[X, Y],$$

where the last equality follows using the torsion free assumption on the connection.

4.
$$\begin{aligned} R(X,Y)Z &= -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z \\ &= -\frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[X,Z]] + -\frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}[X,[Y,Z]] \end{aligned}$$

where the last step relies on the Jacobi identity.

5. $K(X,Y) = \langle R(X,Y)X, Y\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle$ by bi-invariance of the metric.