7. Curvature, conjugate points

7.1. Codazzi equation.

Let $M \subset \bar{M}$ be a submanifold of the Riemannian manifold $(\bar{M}, \bar{g})$. For the second fundamental form K of M, we define

$$(D_X^\perp K)(Y, W) := (\bar{D}_X(K(Y, W))^\perp - K(D_X Y, W) - K(Y, D_X W),$$

where $W, X, Y \in \Gamma(TM)$. Show that the Codazzi equation

$$\left(\bar{R}(X, Y)W\right)^\perp = (D_X^\perp K)(Y, W) - (D_Y^\perp K)(X, W)$$

holds for all $W, X, Y \in \Gamma(TM)$.

7.2. Sectional curvature of submanifolds.

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with sectional curvature $\bar{K}$. Let $p \in \bar{M}$ and $L \subset T\bar{M}_p$ an m-dimensional linear subspace.

1. Prove that there is some $r > 0$ such that for the open ball $B_r(0) \subset T\bar{M}_p$, the set $M := \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\bar{M}$.
2. Let g be the induced metric on M and let K be the sectional curvature of M. Show that for $E \subset TM_p$, we have $K_p(E) = \bar{K}_p(E)$ and if L is a 2-dimensional subspace, then $K \leq \bar{K}$ on M.

7.3. Small balls and scalar curvature.

Let p be a point in the m-dimensional Riemannian manifold (M, g). The goal is to prove the following Taylor expansion of the volume of the ball $B_r(p)$ as a function of r:

$$\mathrm{vol}(B_r(p)) = \Omega_m r^m \left(1 - \frac{1}{6(m+2)}\mathrm{scal}(p) r^2 + O(r^3)\right).$$

1. Let $v \in TM_p$ with $|v| = 1$, define the geodesic $c(t) := \exp_p(tv)$ and let $e_1 = v, e_2, \ldots, e_m \in TM_p$ be an orthonormal basis. Consider the Jacobi fields Y_i along c with $Y_i(0) = 0$ and $\dot{Y}_i(0) = e_i$ for $i = 2, \ldots m$. Show that the volume distortion factor of $\exp_p$ at tv is given by

$$J(v, t) := \sqrt{\det\left(\langle T_{tv} e_i, T_{tv} e_j\rangle\right)} = t^{-(m-1)} \sqrt{\det\left(\langle Y_i, Y_j\rangle\right)},$$

where $T_{tv} \coloneqq (d\exp_p)_{tv}$.

2. Let $E_2, \dots, E_m$ be parallel vector fields along c with $E_i(0) = e_i$. Then the Taylor expansion of Y_i is

$$Y_i(t) = tE_i - \sum_{k=2}^{m} \left(\frac{t^3}{6} R(e_i, v, e_k, v) + O(t^4) \right) E_k.$$

3. Conclude that $J(v,t) = 1 - \frac{t^2}{6}\mathrm{Ric}(v,v) + O(t^4)$.

   Hint: Use $\det(I_m + \epsilon A) = 1 + \epsilon \mathrm{trace}(A) + O(\epsilon^2)$.

4. Prove the above formula for $\mathrm{vol}(B_r(p))$.

7. Solutions

Solution of 7.1: As $\bar{D}_Z W = D_Z W + h(Z,W)$ for $W, Z \in \Gamma(TM)$, we get

$$\begin{aligned}
\bar{R}(X,Y)W &= \bar{D}_X \bar{D}_Y W - \bar{D}_Y \bar{D}_X W - \bar{D}_{[X,Y]} W \\
&= \bar{D}_X (D_Y W + h(Y,W)) - \bar{D}_Y (D_X W + h(X,W)) \\
&\quad - (D_{[X,Y]} W + h([X,Y],W)) \\
&= D_X D_Y W + h(X, D_Y W) + \bar{D}_X (h(Y,W)) \\
&\quad - D_Y D_X W - h(Y, D_X W) - \bar{D}_Y (h(X,W)) \\
&\quad - D_{[X,Y]} W - h(D_X Y - D_Y X, W) \\
&= R(X,Y)W \\
&\quad + \bar{D}_X (h(Y,W)) - h(D_X Y, W) - h(Y, D_X W) \\
&\quad - \bar{D}_Y (h(X,W)) + h(D_Y X, W) + h(X, D_Y W).
\end{aligned}$$

Note that we used that D is torsion free, i.e. $[X,Y] = D_X Y - D_Y X$. Now, taking the normal part, we conclude that the Codazzi equation

$$\left(\bar{R}(X,Y)W\right)^{\perp} = (D_X^{\perp} h)(Y,W) - (D_Y^{\perp} h)(X,W)$$

holds.

Solution of 7.2:

1. First, we know that there is some $r > 0$ such that the restriction of the exponential map to $B_r(0)$, i.e. $\exp_p|_{B_r(0)}\colon B_r(0) \to \exp_p(B_r(0))$, is a diffeomorphism. Furthermore, note that $L \cap B_r(0)$ is an m-dimensional submanifold of $B_r(0)$ and hence $M = \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\exp_p(B_r(0))$. Finally, as $\exp_p(B_r(0))$ is open in $\bar{M}$, it follows that M is a submanifold of $\bar{M}$ as well.

2. Let $u, v \in E$ be an orthonormal basis of $E \subset TM_p$. Then we have

$$\begin{aligned} K_p(E) &= R_p(u,v,u,v) \\ &= \bar{R}_p(u,v,u,v) + \bar{g}_p(h_p(u,u), h_p(v,v)) - \bar{g}_p(h_p(u,v), h_p(u,v)) \\ &= \bar{K}_p(E) + \bar{g}_p(h_p(u,u), h_p(v,v)) - \bar{g}_p(h_p(u,v), h_p(u,v)) \end{aligned}$$

We now prove that $h_p(u,u) = h_p(v,v) = h_p(u,v) = 0$. Extend u, v to an orthonormal basis $e_1 = u, e_2 = v, e_3, \ldots, e_{\bar{m}}$ of $T\bar{M}_p$. Then this basis induces normal coordinates on $\bar{M}$. Hence, we have $\Gamma^k_{ij}(p) = 0$ and thus $(\bar{D}_{e_i} e_j)_p = 0$ for all i, j. In particular this implies that $h_p(u,u) = h_p(v,v) = h_p(u,v) = 0$ as claimed.

Assume now that $L \subset T\bar{M}_p$ is 2-dimensional and let $q := \exp_p(x) \in M$ for $x \in L \cap B_r(0)$. By the above, we may assume that $x \neq 0$.

Define $w := \frac{x}{|x|} \in TM_p$ and let c_w be the unique geodesic with $c(0) = p$ and $\dot{c}(0) = w$. Then we have $q = c_w(|x|)$ and $u := \dot{c}_w(|x|) \in TM_q$ with $|u| = 1$. Furthermore, by the Lemma proven in exercise 6.2, we get

$$h_q(u,u) = \left(\frac{\bar{D}}{dt} \dot{c}_w \Big|_{t=|x|} \right)^{\perp} = 0.$$

To compute the sectional curvature of $E = TM_q$, we extend u to an orthonormal basis u, v of E and get

$$K_q(E) = R_q(u,v,u,v) = \bar{R}_q(u,v,u,v) - |h_q(u,v)|^2 \leq \bar{K}_q(E)$$

as desired.

Solution of 7.3:

1. The Jacobi fields Y_i are given as variation vector fields along c of $\alpha_i(s,t) := \exp_p(t(v+$

$se_i)$), i.e.

$$Y_i(t) = \frac{d}{ds}\bigg|_{s=0} \alpha_i(s,t) = T_{tv}(te_i)$$

and therefore $T_{tv}e_i = \frac{1}{t}Y_i(t)$.

Furthermore, we have $\langle T_{tv}v, T_{tv}e_i\rangle = \langle v, e_i\rangle = 0$ by the Gauss Lemma. Then the volume distortion is given by

$$J(v,t) = \sqrt{\det\Big(\langle T_{tv}e_i, T_{tv}e_j\rangle\Big)} = t^{-(m-1)}\sqrt{\det\Big(\langle Y_i, Y_j\rangle\Big)}.$$

2. We check that the derivatives coincide. Clearly, we have $Y_i(0) = 0$, $\dot{Y}_i(0) = e_i$ and $\ddot{Y}_i(0) = -R(Y_i(0), \dot{c}(0))\dot{c}(0) = 0$. Furthermore,

$$\begin{aligned}\dddot{Y}_i(0) &= -(D_{\dot{c}}R)(Y_i(0), \dot{c}(0))\dot{c}(0) - R(\dot{Y}_i(0), \dot{c}(0))\dot{c}(0)\\ &= -R(e_i, v)v = -\sum_{k=2}^{m}\langle R(e_i,v)v, e_k\rangle e_k = -\sum_{k=2}^{m} R(e_k, v, e_i, v)e_k.\end{aligned}$$

3. With the above, we get

$$\begin{aligned}\langle Y_i, Y_j\rangle &= t^2\langle E_i, E_j\rangle - \tfrac{t^4}{6}\sum_{k=2}^{m} R(e_i, v, e_k, v)\,\langle E_k, E_j\rangle\\ &\qquad\qquad - \tfrac{t^4}{6}\sum_{k=2}^{m} R(e_j, v, e_k, v)\,\langle E_i, E_k\rangle + O(t^5)\\ &= t^2\delta_{ij} - \tfrac{t^4}{3}R(e_i, v, e_j, v) + O(t^5)\end{aligned}$$

and thus

$$\begin{aligned}J(v,t) &= \sqrt{\det\Big(\delta_{ij} - \tfrac{t^2}{3}R(e_i, v, e_j, v) + O(t^3)\Big)}\\ &= \sqrt{1 - \tfrac{t^2}{3}(R(e_i, v, e_j, v)) + O(t^3)}\\ &= 1 - \tfrac{t^2}{6}(v,v) + O(t^3).\end{aligned}$$

4. First, we use polar coordinates:

$$vol(B_r(p)) = \int_{B_r(0)} J(v,t)\,{}^1 \ldots {}^m = \int_0^r \int_{S^{m-1}} t^{m-1}J(v,t)\,{}^{S^{m-1}}.$$

Then, using exercise 2 and the above, we get

$$\begin{aligned}
vol(B_r(p)) &= \int_0^r \int_{S^{m-1}} t^{m-1}(1 - \tfrac{t^2}{6}(v,v) + O(t^3))\, vol^{S^{m-1}} \\
&= \int_0^r t^{m-1} \left(vol(S^{m-1}) - \tfrac{t^2}{6} \int_{S^{m-1}} (v,v)^{\ S^{m-1}} + O(t^3) \right) \\
&= \frac{r^m}{m} m\Omega_m - \frac{r^{m+2}}{6(m+2)} \mathrm{scal}(p)\Omega_m + O(r^{m+3}) \\
&= \Omega_m r^m \left(1 - \frac{r^2}{6(m+2)} \mathrm{scal}(p) + O(r^3) \right).
\end{aligned}$$