8. Isometries, translations, geodesics and conjugate points

8.1. Nearby conjugate points.

Prove the following Lemma.

Suppose $\gamma\colon [0,1] \to M$ is a geodesic and $t_0 \in (0,1)$ is such that $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ. Then there exists $\epsilon > 0$ so that the following holds: if $c\colon [0,1] \to M$ is a geodesic with $d(\gamma(t), c(t)) < \epsilon$ for all $t \in [0,1]$, then there exists $t_1 \in (0,1)$ so that $c(t_1)$ is conjugate to $c(0)$ along c.

8.2. Locally symmetric spaces.

Let M be a connected m-dimensional Riemannian manifold. Then M is called *locally symmetric* if for all $p \in M$ there is a normal neighborhood $B(p,r)$ such that the *local geodesic reflection* $\sigma_p := \exp_p \circ(-\operatorname{id}) \circ \exp_p^{-1}\colon B(p,r) \to B(p,r)$ is an isometry.

1. Show that if M is locally symmetric, then $DR \equiv 0$.
   Hint: Use that $d(\sigma_p)_p = -\operatorname{id}$ on TM_p.
2. Suppose that $DR \equiv 0$. Show that if $c\colon [-1,1] \to M$ is a geodesic and $\{E_i\}_{i=1}^m$ is a parallel orthonormal frame along c, then $R(E_i, c')c' = \sum_{k=1}^m r_i^k E_k$ for *constants* r_i^k.
3. Show that if $DR \equiv 0$, then M is locally symmetric.
   Hint: Let $q \in B(p,r), q \neq p$, and $v \in TM_q$. To show that $|d(\sigma_p)_q(v)| = |v|$, consider the geodesic $c\colon [-1,1] \to B(p,r)$ with $c(0) = p, c(1) = q$, and a Jacobi field Y along c with $Y(0) = 0$ and $Y(1) = v$. Use 2..

8.3. Poincaré models of hyperbolic space.

Let us introduce the following two well-known models of the hyperbolic space:

$$\text{Unit ball } \{|z| < 1\} \subset \mathbb{R}^n \text{ equipped with metric } g_{ij} = \frac{4\delta_{ij}}{(1-|z|^2)^2}$$

and

$$\text{Half space } \{x^n > 0\} \subset \mathbb{R}^n \text{ equipped with metric } g_{ij} = \frac{\delta_{ij}}{(x^n)^2}.$$

1. Show that composing the transformations $y = x + (\frac{1}{2} - 2x^n)\boldsymbol{e}_n$ and $z = \boldsymbol{e}_n + (y - \boldsymbol{e}_n)|y - \boldsymbol{e}_n|^{-2}$ give an isometry between the two previous Riemannian manifolds
2. Show that, for the second model, circular arcs at $\{x^n = 0\}$ are geodesics.

3. Show that given any given point all geodesic rays $x(t)$, $t \geq 0$ emanating from it are minimizing up to arbitrarily large values of $t > 0$ (note that this is stronger than geodesic completeness).

4. Show that the sectional curvatures are constantly equal to -1.

8.4. Translations.
Suppose that Γ is a group of translations of $\mathbb{R}^m$ that acts freely and properly discontinuously on $\mathbb{R}^m$.

1. Show that there exist linearly independent vectors $v_1, \dots, v_k \in \mathbb{R}^m$ such that

$$\Gamma = \left\{ x \mapsto x + \sum_{i=1}^{k} z_i v_i : (z_1, \dots, z_k) \in \mathbb{Z}^k \right\} \simeq \mathbb{Z}^k.$$

2. Let l denote the infimum of the lengths of all closed curves in $\mathbb{R}^m/\Gamma$ that are not null-homotopic. Show that l equals the length of the shortest non-zero vector of the form $\sum_{i=1}^{k} z_i v_i$ with $z_i \in \mathbb{Z}$ as above.

8. Solutions

Solution of 8.1: Recall that for a (constant speed) geodesic $\gamma : [0,1] \to M$, there exists $0 < t_* < 1$ such that $\gamma(t_*)$ is conjugate to $\gamma(0)$ along γ if and only if the second variation of length is negative along some variation vector field which vanishes near the endpoints. Indeed, if such a point exists, then the proof of Theorem 6.12 exhibits such a vector field; on the other hand, suppose that there are no conjugate points in $(0,1)$. Then, given a variation vector field X that vanishes for $t \geq 1 - 2\delta$, with some $\delta > 0$, since there are no conjugate points of γ in the interval $(0, 1-\delta]$, by Theorem 6.8 γ is locally minimizing there, and in particular the second variation of the proper variation vector field X is nonnegative.

With this observation in hand, the exercise is about making precise that this condition is “open”. More precisely, since γ has a conjugate point $\gamma(t_0)$ with $0 < t_0 < 1$, there exists a piecewise smooth vector field X along γ, vanishing near 0 and 1, with negative second variation. Let us consider a small contractible neighborhood U_1 of $(p, v) = (\gamma(0), \gamma'(0))$ in TM, let $\phi : U_1 \times [0,1]$ be the geodesic flow on this set (that is, $\phi(q, w, \cdot)$ is the geodesic with initial data (q, w)), and extend X to a piecewise smooth vector field $\widetilde{X}$ along ϕ

vanishing near $U_1 \times \{0,1\}$ (we can do this by choosing a frame of TM along ϕ and extending componentwise). Since ϕ is piecewise smooth, the function

$$(q,w) \in U_1 \mapsto I_{\phi(q,w,\cdot)}(\widetilde{X},\widetilde{X}) = \int_0^1 |(\partial_t \widetilde{X})^\perp|^2 - R(\widetilde{X}, \partial_t\phi, \partial_t\phi, \widetilde{X})\, dt$$

is continuous, and since it is negative at (p,v) it must also be negative in a neighborhood U of (p,v).

Finally we need to show that for this neighborhood $U \subset TM$ of the initial data $(p,v) \in TM$ there exists $\epsilon > 0$ such that, if a geodesic $c : [0,1] \to M$ satisfies $d(\gamma(t), c(t)) < \epsilon$ for all $t \in [0,1]$, then the initial data $(c(0), c'(0))$ lies in U. We do this by contradiction: suppose that no such ϵ exists, thus we can find a sequence of geodesics $c_j : [0,1] \to M$ such that $d(\gamma(t), c_j(t)) < j^{-1}$ but $(c_j(0), c_j'(0)) \notin U$.

First we need to show that the lengths of the curves are controlled. For that, let $a > 0$ be small enough so that every geodesic contained in $B_a(p)$ is length-minimizing. Then let $t = \frac{a}{2|v|}$ and choose $j > \frac{2}{a}$, so that $\gamma([0,t])$ is contained in $\overline{B}_{a/2}(p)$ and thus $c_j([0,t])$ is contained in $B_a(p)$. Then

$$\begin{aligned} t|c_j'(0)| &= L(c_j([0,t])) = d(c_j(0), c_j(t)) \\ &\leq d(c_j(0), \gamma(0)) + d(\gamma(0), \gamma(t)) + d(\gamma(t), c_j(t)) \\ &\leq \frac{2}{j} + t|v|, \end{aligned}$$

hence $|c_j'(0)| \leq |v| + \frac{4|v|}{ja}$ is bounded independently of j. Note also that $d(c(0), p) \leq j^{-1}$, so $c_j(0) \to p$. Hence, after taking a subsequence, $(c_j(0), c_j'(0)) \to (p, w)$ for some $w \in T_pM$. If $\widetilde{\gamma}$ denotes the geodesic with $\widetilde{\gamma}(0) = p$ and $\widetilde{\gamma}'(0) = w$, then by the theorem of smooth dependence on initial data for ODEs, we have that $c_j \to \widetilde{\gamma}$ uniformly. But also $c_j \to \gamma$ uniformly, hence $\gamma = \widetilde{\gamma}$, which implies that $(p,v) = (p,w) = \lim_{j\to\infty}(c_j(0), c_j'(0))$ and thus $(c_j(0), c_j'(0)) \in U$ for j large enough.

Solution of 8.2: (a) Suppose that M is locally symmetric, let $p \in M$ and $w, x, y, z \in TM_p$.

Then, since σ_p is an isometry and $d(\sigma_p)_p = -\,\mathrm{id}$ on TM_p we have

$$\begin{aligned}-(D_w R)(x,y)z &= d(\sigma_p)_p (D_w R)(x,y)z \\ &= (D_{d(\sigma_p)_p w})(d(\sigma_p)_p x, d(\sigma_p)_p y) d(\sigma_p)_p z \\ &= (D_{-w}R)(-x,-y) - z \\ &= (D_w R)(x,y)z,\end{aligned}$$

so $(D_w R)(x,y)z = 0$.

b) Recall that for $X, Y, Z, W \in \Gamma(TM)$

$$\begin{aligned}D_W(R(X,Y)Z) =& R(X,Y)D_W(Z) + R(D_W X, Y) \\ &+ R(X, D_W Y)Z + (D_W R)(X,Y)Z.\end{aligned}$$

Now, write $R(E_i, c')c' = \sum_{k=1}^m f_i^k E_k$ for some functions $f_i^k\colon [-1,1] \to \mathbb{R}$. Since E_i and c' are parallel vector fields, the above relation implies that

$$\begin{aligned}0 &= (D_{\partial/\partial t} R)(E_i, c')c' \\ &= D_{\partial/\partial t}\Big(R(E_i, c')c'\Big) \\ &= \sum_{k=1}^m D_{\partial/\partial t}(f_i^k E_k) \\ &= \sum_{k=1}^m \Big(\dot{f}_i^k E_k + f_i^k D_{\partial/\partial t} E_k\Big) \\ &= \sum_{k=1}^m \dot{f}_i^k E_k,\end{aligned}$$

hence the f_i^k are constant.

c) Let $q \in B(p,r)$, $q \neq p$ and $v \in TM_q$. We must show that $|d(\sigma_p)_q(v)| = |v|$. Let $c\colon [-1,1] \to M$ be the geodesic with $c(0) = p$ and $c(1) = q$. Let Y be the Jacobi field along c with $Y(0) = 0$ and $Y(1) = v$. Since σ_p reverts geodesics it follows that $d(\sigma_p)_q Y(1) = Y(-1)$, so it remains to show that $|Y(1)| = |Y(-1)|$. Write $Y = \sum_{i=1}^m h^i E_i$ for some functions $h^i\colon [-1,1] \to \mathbb{R}$ then the Jacobi equation implies that

$$\ddot{h}^k + \sum_{i=1}^m h^i r_i^k = 0,$$

with $h^i(0) = 0$, for $k = 1, \dots, m$. It follows that $h^i(-t) = -h^i(t)$ for all $t \in [-1,1]$. In particular $|Y(-1)| = |Y(1)|$.

Solution of 8.3: a) We have

$$dz = (y - \boldsymbol{e}_n)|y - \boldsymbol{e}_n|^{-2}dy - 2|y - \boldsymbol{e}_n|^{-4}(y - \boldsymbol{e}_n) \cdot dy(y - \boldsymbol{e}_n),$$

$$|dz|^2 = |y - \boldsymbol{e}_n|^{-4}|dy|^2$$

$$1 - |z|^2 = (1 - 2y^n)|y - \boldsymbol{e}_n|^{-2}$$

Hence, using $|dy| = |dx|$ and $2y^n - 1 = -2x^n$ we obtain

$$\frac{4|dz|^2}{(1 - |z|^2)^2} = \frac{4|dy|^2}{(1 - 2y^n)^2} = \frac{|dx|^2}{(x^n)^2}$$

b) In order to compute the geodesic equation we let $x_(t) := x(t) + \xi(t)$, where both x, ξ are function from (a, b) to $\{x^n > 0\}$, ξ vanishing at a and b. We have

$$0 = \tfrac{d}{d}\Big|_{=0} L(x_) = \tfrac{d}{d}\Big|_{=0} \int_a^b \frac{|x' + \xi'|}{(x^n + \xi^n)} dt = \int_a^b \frac{x' \cdot \xi'}{|x'|(x^n)} - \frac{|x'|}{(x^n)^2}\xi^n dt.$$

After integrating by parts and using that ξ is arbitrary we find

$$-\left(\frac{x'}{|x'|(x^n)}\right)' - \frac{|x'|}{(x^n)^2}\boldsymbol{e}_n = 0.$$

Also, $x(t)$ is parametrized by the arc length iff $\frac{|x'(t)|^2}{(x^n(t))^2} = 1$.

Hence, we obtain

$$\left(\frac{(x^\alpha)'}{(x^n)^2}\right)' = 0 \quad \text{for } \alpha = 1, 2, \ldots, n-1 \qquad \left(\frac{(x^n)'}{(x^n)^2}\right)' + \frac{1}{x^n} = 0$$

Take now $x(t) = R\cos\theta(t)\boldsymbol{e}_1 + R\sin\theta(t)\boldsymbol{e}_n$, for some $R > 0$, with $\theta(t)$ satisfying $\theta' = \sin\theta$.

We have:

$$\left(\frac{(x^1)'}{(x^n)^2}\right)' = \left(\frac{-\sin\theta\theta'}{R\sin^2\theta}\right)' + \frac{1}{R\sin\theta} = \big(-1/R\big)' = 0$$

and

$$\begin{aligned}\left(\frac{(x^n)'}{(x^n)^2}\right)' + \frac{1}{x^n} &= \left(\frac{\cos\theta\theta'}{R\sin^2\theta}\right)' + \frac{1}{R\sin\theta} \\ &= \frac{(\operatorname{cotan}\theta)'}{R} + \frac{1}{R\sin\theta} = \frac{-\theta'}{R\sin^2\theta} + \frac{1}{R\sin\theta} = 0.\end{aligned}$$

Hence (using that the metric is invariant under translations and rotation in the first $n-1$ variables, we have shown that half circular arcs with centers on $\{x^n = 0\}$ are geodesics. Since for any point $p \in \{x^n > 0\}$ and for any unit vector $v \in \mathbb{S}^{n-1}$ there is a (unique) half circular arc with center on $\{x^n = 0\}$ through p and tangent to v, these are *all* geodesics.

c) The geodesic completeness follows from the fact that $\theta(t)$ above (satisfying $\theta' = \sin\theta$) is the arc length and $\int_a^b \frac{d\theta}{\sin\theta} \to +\infty$ if $a \downarrow 0$ or $b \uparrow \pi$. Also, since given any two points in$\{x^n > 0\}$ there is a unique half circular arc with center on $\{x^n = 0\}$ through them, this must be the minimizing geodesic joining them. As a consequence, any geodesic joining any two points is minimizing.

d) By Koszul's formula, for $\alpha, \beta = 1, 2, \dots, n-1$ we have

$$\nabla_{\partial_\alpha}\partial_\beta = (x^n)^{-1}\delta_{\alpha\beta}\partial_n, \quad \nabla_{\partial_n}\partial_\alpha = (x^n)^{-1}\partial_\alpha, \quad \nabla_{\partial_n}\partial_n = (x^n)^{-1}\partial_n$$

Hence,

$$\nabla_{\partial_\beta}\nabla_{\partial_\alpha}\partial_\beta = -(x^n)^2\delta_{\alpha\beta}\partial_\beta, \qquad \nabla_{\partial_\alpha}\nabla_{\partial_\beta}\partial_\beta = -(x^n)^{-2}\partial_\alpha.$$

This implies

$$R(\partial_\beta, \partial_\alpha)\partial_\beta = -(x^n)^2\partial_\alpha.$$

Similarly,

$$R(\partial_n, \partial_\alpha)\partial_n = -(x^n)^2\partial_\alpha.$$

This implies that the sectional curvatures are constantly equal to -1.

Solution of 8.4: a) For each $g \in \Gamma$ there is some $v_g \in \mathbb{R}^m$ such that $gx = x + v_g$ for all $x \in \mathbb{R}^m$ and since Γ acts freely, we have $v_g \neq 0$ for $g \neq \mathrm{id}$. We denote $V := \{v_g \in \mathbb{R}^m : g \in \Gamma\}$. Note that, as Γ acts properly discontinuously, $V \cap B_r(0)$ is finite for all $r > 0$ and thus each subset of V has an element of minimal length.

We now do induction on m. For $m = 1$, choose $g \in \Gamma \setminus \{\mathrm{id}\}$ such that $|v_g|$ is of minimal length. If there is some $v \in V$ with $v = \lambda v_g$, $\lambda \notin \mathbb{Z}$, we also have $w := v - \lfloor\lambda\rfloor v_g \in V \setminus \{0\}$ with $|w| < |v_g|$, a contradiction to minimality.

For $m \geq 2$, let $v_g \in V \setminus \{0\}$ be of minimal length and let $V' := \mathrm{span}(v_g) \cap V$. By the same argument as above, we get $V' = \mathbb{Z}v_g$.

Then we have $\mathbb{R}^m = \mathbb{R}^{m-1} \oplus \mathbb{R}v_g$ with projection map $\pi\colon \mathbb{R}^m \to \mathbb{R}^{m-1}$ and $\Gamma' := \Gamma/g\mathbb{Z}$ acts by translations on $\mathbb{R}^{m-1}$ via $[h]x = x + \pi(v_h)$. As for $h \notin g\mathbb{Z}$ we have $\pi(v_h) \neq 0$, this action is free. We claim that it is properly discontinuous as well. If not, there are $(h_n)_{n\in\mathbb{N}} \in \Gamma$ with $\pi(v_{h_n}) \neq \pi(v_{h_{n'}})$ and $|\pi(v_{h_n})| < r$ for some $r > 0$. But then, there are $l_n \in \mathbb{Z}$ such that $|v_{h_n} - \pi(v_{h_n}) - l_n v_g| < |v_g|$, i.e. $(v_{h_n - l_n g})_{n\in N}$ is an infinite subset of $V \cap B_{r+|v_g|}(0)$, contradicting that Γ acts properly discontinuously.

By our induction hypothesis, there are $h_2, \dots, h_k \in \Gamma$ such that

$$\pi(V) = \mathbb{Z}\pi(v_{h_2}) \oplus \ldots \oplus \mathbb{Z}\pi(v_{h_k})$$

and consequently $V = \mathbb{Z}v_g \oplus \mathbb{Z}v_{h_2} \oplus \ldots \oplus \mathbb{Z}v_{h_k}$.

b) Let $\pi\colon \mathbb{R}^m \to \mathbb{R}^m/\Gamma$ denote the covering map and let $c\colon [0,1] \to \mathbb{R}^m/\Gamma$ be a closed curve in $\mathbb{R}^m/\Gamma$. Then for $p \in \pi^{-1}(c(0))$, there exists a unique lift $\overline{c}\colon [0,1] \to \mathbb{R}^m$ of c with $\overline{c}(0) = p$. Furthermore, if c is not null-homotopic, we have $q := \overline{c}(1) \neq \overline{c}(0)$ and therefore

$$L(c) = L(\overline{c}) \geq d(p,q) = \left|\sum_{i=1}^k z_i v_i\right|,$$

for some $(z_1, \dots, z_k) \in \mathbb{Z}^k \setminus \{0\}$.

Finally, if $v = \sum_{i=1}^k z_i v_i \neq 0$ is of minimal length, then $c\colon [0,1] \to \mathbb{R}^m/\Gamma$, $c(t) := \pi(tv)$, has length $L(c) = |v|$.