Introduction to Mathematical Finance

Exercise sheet 1

Please submit your solutions online until Wednesday 22:00, 28/02/2024.

Exercise 1.1 Let $\mathcal{C} := \mathbb{R} \times \mathbb{R}^K$ be the consumption space, $\mathcal{D}$ the payoff matrix, e^i an endowment and π a price vector. Recall the budget set

$$B(e^i, \pi) := \{c \in \mathcal{C} : \exists \vartheta \in \mathbb{R}^N \text{ with } c_0 \leq e^i_0 - \vartheta \cdot \pi \text{ and } c_T \leq e^i_T + \mathcal{D}\vartheta\}.$$

(a) For the statements

1. $c \in B(e^i, \pi)$,
2. $c - e^i \in B(0, \pi)$,
3. $c - e^i$ is attainable from 0 initial wealth endowment,

show that (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3).

(b) Show by an example that (2) $\Rightarrow$ (3) is not true in general.

Exercise 1.2 Let $\succeq$ be a preference order on $\mathcal{C}$ satisfying axioms (P1)-(P5). A function $\mathcal{U} : \mathcal{C} \to \mathbb{R}$ is called a *utility functional representing* $\succeq$ or *a numerical representation of* $\succeq$ if

$$c' \succeq c \iff \mathcal{U}(c') \geq \mathcal{U}(c).$$

(a) Show that all $\mathcal{U}$ representing $\succeq$ must be *quasiconcave*, i.e., for all $c, c' \in \mathcal{C}$ and $\lambda \in [0, 1]$,

$$\mathcal{U}\Big(\lambda c + (1 - \lambda)c'\Big) \geq \min\{\mathcal{U}(c), \mathcal{U}(c')\}.$$

(b) Which axioms are needed for this result?

(c) Show by a counterexample that a preference order can be represented by a utility functional which is not concave.

Exercise 1.3

(a) Construct a market with arbitrage of the first kind but with no arbitrage of the second kind.

(b) Construct a market with arbitrage of the second kind but with no arbitrage of the first kind.

(c) Find a sufficient condition under which existence of an arbitrage of the second kind implies the existence of an arbitrage of the first kind.