Introduction to Mathematical Finance

Exercise sheet 10

Please submit your solutions online until Wednesday 10pm, 08/05/2024.

Exercise 10.1

(a) Let (ϑ, c) be a self-financing investment and consumption pair and

$$W_t = W_t^{v_0,\vartheta,c} = v_0 + \sum_{j=1}^{t}(\vartheta_j \cdot \triangle X_j - c_{j-1})$$

for $t = 0, 1, \ldots, T$ the corresponding discounted wealth process. Show that if $W \geq -a$ for some constant a, then W is a Q-supermartingale for any ELMM Q for X.

(b) Let $U : \mathbb{R} \to \mathbb{R}$ be concave and consider for fixed $Q \in \mathbb{P}_{\mathrm{loc}}$ the problem of maximising $E_Q[U(W_T^{v_0,\vartheta,c} - c_T)]$ over all self-financing investment and consumption pairs. Assuming that each $U(W_\cdot^{v_0,\vartheta,c})$ is Q-integrable and that $j_0 := \sup_{(\vartheta,c)} E_Q[U(W_T^{v_0,\vartheta,c} - c_T)] < \infty$, show that the solution is given by $\vartheta \equiv 0$, $c \equiv 0$.

Exercise 10.2 (Mean-variance hedging). Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\ldots,T}$. Suppose that the discounted price process X satisfies $E_Q[(\triangle X_t)^2|\mathcal{F}_{t-1}] < \infty$ P-a.s. for all t. Define

$$\mathcal{A} := \{\text{all predictable processes } \vartheta = (\vartheta_t)_{t=1,\ldots,T} : (\vartheta \cdot X)_t \in L^2 \text{ for } t = 1, \ldots, T\}.$$

Let $c \in \mathbb{R}$ and $H \in L^2(\mathcal{F}_T, P)$. Consider the problem

$$\min_{\vartheta \in \mathcal{A}} E\left[\Big(H - c - (\vartheta \cdot X)_T\Big)^2\right].$$

The goal of this exercise is to construct a candidate for an optimal strategy using the MOP. For $\vartheta \in \mathcal{A}$, we set

$$\begin{aligned}\mathcal{A}_t(\vartheta) &:= \{\vartheta' \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j \leq t\},\\ \mathcal{A}_t &:= \mathcal{A}_t(0) = \{\vartheta \in \mathcal{A} : \vartheta_j = 0 \text{ for } j \leq t\}.\end{aligned}$$

For $v_t \in L^2(\mathcal{F}_t)$, we define

$$\begin{aligned}\Gamma_t(v_t, \vartheta') &:= E\Big[\Big(H - v_t - \sum_{j=t+1}^{T} \vartheta'_j \triangle X_j\Big)^2 \Big| \mathcal{F}_t\Big],\\ Y_t(v_t) &:= \operatorname{ess\,inf}_{\vartheta' \in \mathcal{A}_t} \Gamma_t(v_t, \vartheta').\end{aligned}$$

(a) Show that for each t and each $v_t \in L^2(\mathcal{F}_t)$, the collection of random variables

$$\Lambda_t(v_t) := \{\Gamma_t(v_t, \vartheta') : \vartheta' \in \mathcal{A}_t\}$$

is closed under taking minima.

(b) Show that for fixed $\vartheta \in \mathcal{A}$, $x \in \mathbb{R}$, the process $(Y_t(x + (\vartheta \cdot X)_t))_{t=0,\ldots,T}$ is a submartingale.

(c) Show that $\vartheta^* \in \mathcal{A}$ is optimal if and only if the process $(Y_t(c + (\vartheta^* \cdot X)_t))_{k=0,\ldots,T}$ is a martingale.

(d) Show that $(Y_t(x))$ satisfies the recursion

$$Y_{t-1}(x) = \operatorname{ess\,inf}_{\vartheta' \in \mathcal{A}_{t-1}} E[Y_t(x + \vartheta'_t \triangle X_t)|\mathcal{F}_{t-1}]$$

with $Y_T(x) = (H - x)^2$.

Exercise 10.3 Consider a general arbitrage-free single-period market with $\mathcal{F}_0$ trivial. Fix x and let $U : (0, \infty) \to \mathbb{R}$ be a concave, increasing, continuously differentiable (utility) function such that

$$\sup_{\vartheta \in \mathcal{A}(x)} E[U(x + \vartheta \cdot \triangle X_1)] < \infty, \tag{1}$$

with

$$\mathcal{A}(x) = \{\vartheta \in \mathbb{R}^d : x + \vartheta \cdot \triangle X_1 \geq 0 \ P\text{-a.s.}, U(x + \vartheta \cdot \Delta X_1) \in L^1\}.$$

Furthermore, assume that the supremum is attained in an interior point ϑ^* of $\mathcal{A}(x)$.

Show that we have the *first order condition*

$$E[U'(x + \vartheta^* \cdot \triangle X_1)\triangle X_1] = 0.$$

Hint: You may use that due to concavity,

$$y \mapsto \frac{U(y) - U(z)}{y - z}, \quad y \in (0, \infty) \setminus \{z\}$$

is nonincreasing. By optimality, ϑ^* is better than $\vartheta^* + \varepsilon\eta$ for any $\eta \neq 0$ and $0 < \varepsilon \ll 1$; so take the difference of the corresponding utilities, divide by ε and look at $\varepsilon \searrow 0$. Exploit the hint to see that this quantity is monotonic in ε.