Introduction to Mathematical Finance

Exercise sheet 10

Please submit your solutions online until Wednesday 10pm, 08/05/2024.

Exercise 10.1

(a) Let (ϑ, c) be a self-financing investment and consumption pair and

$$W_t = W_t^{v_0,\vartheta,c} = v_0 + \sum_{j=1}^{t}(\vartheta_j \cdot \triangle X_j - c_{j-1})$$

for $t = 0, 1, \dots, T$ the corresponding discounted wealth process. Show that if $W \geq -a$ for some constant a, then W is a Q-supermartingale for any ELMM Q for X.

(b) Let $U : \mathbb{R} \to \mathbb{R}$ be concave and consider for fixed $Q \in \mathbb{P}_{\text{loc}}$ the problem of maximising $E_Q[U(W_T^{v_0,\vartheta,c} - c_T)]$ over all self-financing investment and consumption pairs. Assuming that each $U(W_\cdot^{v_0,\vartheta,c})$ is Q-integrable and that $j_0 := \sup_{(\vartheta,c)} E_Q[U(W_T^{v_0,\vartheta,c} - c_T)] < \infty$, show that the solution is given by $\vartheta \equiv 0$, $c \equiv 0$.

Solution 10.1

(a) This is just Lemma II.6.2. Indeed, without loss of generality assume $a \geq 0$. Write $C_k := \sum_{j=0}^{k-1} c_j$. First note that $W_k + C_k = v_0 + (\vartheta \cdot X)_k$. By Proposition C.4, $W + C$ is a Q-local martingale. Since $W \geq -a$ and $C \geq 0$, we have that $W + C \geq -a$ is thus a Q-supermartingale and Q-integrable. So $C = W + C - W \leq W + C + a$ is Q-integrable. Then $W = W + C - C$ is Q-integrable and then a Q-supermartingale like $W + C$ since C is increasing.

(b) First we observe as in the lecture that

$$J_T(0,0) = \operatorname{ess\,sup}_{c'_T} U(v_0 - c'_T)$$

and $E_Q[J_T(0,0)] \geq \sup_{c'_T} E_Q[U(v_0 - c'_T)] \geq E_Q[U(v_0)]$. On the other hand, using that $W^{v_0,\vartheta,c}$ is Q-supermartingale and U is concave with $U(W_\cdot^{v_0,\vartheta,c}) \in L^1(Q)$ yields that $U(W_\cdot^{v_0,\vartheta,c} - c_T)$ is a Q-supermartingale for any $(\vartheta, c) \in \mathcal{A}$. Thus $U(v_0) \geq \operatorname{ess\,sup}_{(\vartheta',c') \in \mathcal{A}} E_Q[U(W_T^{v_0,\vartheta',c'} - c'_T)|\mathcal{F}_0]$ and

$$E_Q[U(v_0)] \geq E_Q[J_0(0,0)] \geq E_Q[J_T(0,0)] \geq E_Q[U(v_0)].$$

This implies that $J(0,0)$ is a supermartingale with constant expectation j_0, thus a martingale. Therefore, by the Martingale Optimality Principle: $\vartheta \equiv 0$, $c \equiv 0$ is optimal.

Exercise 10.2 (Mean-variance hedging). Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\dots,T}$. Suppose that the discounted price process X satisfies $E_Q[(\triangle X_t)^2 | \mathcal{F}_{t-1}] < \infty$ P-a.s. for all t. Define

$$\mathcal{A} := \{\text{all predictable processes } \vartheta = (\vartheta_t)_{t=1,\dots,T} : (\vartheta \cdot X)_t \in L^2 \text{ for } t = 1, \dots, T\}.$$

Let $c \in \mathbb{R}$ and $H \in L^2(\mathcal{F}_T, P)$. Consider the problem

$$\min_{\vartheta \in \mathcal{A}} E\left[\left(H - c - (\vartheta \cdot X)_T\right)^2\right].$$

The goal of this exercise is to construct a candidate for an optimal strategy using the MOP. For $\vartheta \in \mathcal{A}$, we set

$$\begin{aligned} \mathcal{A}_t(\vartheta) &:= \{\vartheta' \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j \le t\}, \\ \mathcal{A}_t &:= \mathcal{A}_t(0) = \{\vartheta \in \mathcal{A} : \vartheta_j = 0 \text{ for } j \le t\}. \end{aligned}$$

For $v_t \in L^2(\mathcal{F}_t)$, we define

$$\begin{aligned} \Gamma_t(v_t, \vartheta') &:= E\Big[\Big(H - v_t - \sum_{j=t+1}^{T} \vartheta'_j \triangle X_j\Big)^2 \Big| \mathcal{F}_t\Big], \\ Y_t(v_t) &:= \operatorname{ess\,inf}_{\vartheta' \in \mathcal{A}_t} \Gamma_t(v_t, \vartheta'). \end{aligned}$$

(a) Show that for each t and each $v_t \in L^2(\mathcal{F}_t)$, the collection of random variables

$$\Lambda_t(v_t) := \{\Gamma_t(v_t, \vartheta') : \vartheta' \in \mathcal{A}_t\}$$

is closed under taking minima.

(b) Show that for fixed $\vartheta \in \mathcal{A}$, $x \in \mathbb{R}$, the process $(Y_t(x + (\vartheta \cdot X)_t))_{t=0,\dots,T}$ is a submartingale.

(c) Show that $\vartheta^* \in \mathcal{A}$ is optimal if and only if the process $(Y_t(c + (\vartheta^* \cdot X)_t))_{k=0,\dots,T}$ is a martingale.

(d) Show that $(Y_t(x))$ satisfies the recursion

$$Y_{t-1}(x) = \operatorname{ess\,inf}_{\vartheta' \in \mathcal{A}_{t-1}} E[Y_t(x + \vartheta'_t \triangle X_t) | \mathcal{F}_{t-1}]$$

with $Y_T(x) = (H - x)^2$.

Solution 10.2

(a) Let $\vartheta^1, \vartheta^2 \in \mathcal{A}_k$. Define

$$\vartheta^3 := \vartheta^1 I_A + \vartheta^2 I_{A^c},$$

where $A := \{\Gamma_k(v_k, \vartheta^1) \leq \Gamma_k(v_k, \vartheta^2)\}$. This gives $\vartheta^3_j = 0$ for $j \leq k$. Using that $(\vartheta^i \cdot X)_n \in L^2$ for $i \in \{1, 2\}$ and $(\vartheta^3 \cdot X)_n = I_A(\vartheta^1 \cdot X)_n + I_{A^c}(\vartheta^2 \cdot X)_n$, we have $(\vartheta^3 \cdot X)_n \in L^2(\mathcal{F}_n)$ for each n. Thus

$$
\begin{aligned}
H - v_k - \sum_{j=k+1}^{T} \vartheta^3_j \triangle X_j = I_A\Big(H - v_k - \sum_{j=k+1}^{T} \vartheta^1_j \triangle X_j\Big) \\
+ I_{A^c}\Big(H - v_k - \sum_{j=k+1}^{T} \vartheta^2_j \triangle X_j\Big)
\end{aligned}
$$

is also in L^2. Finally, since $A \in \mathcal{F}_k$, we obtain

$$
\begin{aligned}
\Gamma_k(v_k, \vartheta^3) &= E\Big[\Big(H - v_k - \sum_{j=k+1}^{T} \vartheta^3_j \triangle X_j\Big)^2 \Big| \mathcal{F}_k\Big] \\
&= I_A \Gamma_k(v_k, \vartheta^1) + I_{A^c} \Gamma_k(v_k, \vartheta^2) \\
&= \min\{\Gamma_k(v_k, \vartheta^1), \Gamma_k(v_k, \vartheta^2)\}.
\end{aligned}
$$

(b) Fix $k \leq \ell$. We apply part (a) with $v_k = x + (\vartheta \cdot X)_k \in L^2(\mathcal{F}_k)$. So Corollary E.2 yields

$$
\begin{aligned}
Y_\ell\Big(x + (\vartheta \cdot X)_\ell\Big) &= \operatorname{ess\,inf}_{\vartheta' \in \mathcal{A}_\ell} \Gamma_\ell\Big(x + (\vartheta \cdot X)_\ell, \vartheta'\Big) \\
&=\downarrow \lim_{n \to \infty} E\Big[\Big(H - x - \sum_{j=1}^{\ell} \vartheta_j \triangle X_j - \sum_{j=\ell+1}^{T} \vartheta^n_j \triangle X_j\Big)^2 \Big| \mathcal{F}_\ell\Big]
\end{aligned}
$$

for a sequence $(\vartheta^n)_{n \in \mathbb{N}} \subseteq \mathcal{A}_\ell \subseteq \mathcal{A}_k$. Note that all $\Gamma_\ell(x + (\vartheta \cdot X)_\ell, \vartheta^n)$ are in L^1 due to the definitions of $\vartheta, (\vartheta^n)_{n \in \mathbb{N}}$. Then using monotone convergence and the tower property, we have

$$
\begin{aligned}
E\Big[Y_\ell(x + (\vartheta \cdot X)_\ell) \Big| \mathcal{F}_k\Big] &= E\Big[\lim_{n \to \infty} E\Big[\Big(H - x - \sum_{j=1}^{\ell} \vartheta_j \triangle X_j - \sum_{j=\ell+1}^{T} \vartheta^n_j \triangle X_j\Big)^2 \Big| \mathcal{F}_\ell\Big] \Big| \mathcal{F}_k\Big] \\
&= \lim_{n \to \infty} E\Big[E\Big[\Big(H - x - \sum_{j=1}^{\ell} \vartheta_j \triangle X_j - \sum_{j=\ell+1}^{T} \vartheta^n_j \triangle X_j\Big)^2 \Big| \mathcal{F}_\ell\Big] \Big| \mathcal{F}_k\Big] \\
&= \lim_{n \to \infty} E\Big[\Big(H - x - \sum_{j=1}^{k} \vartheta_j \triangle X_j - \sum_{j=k+1}^{T} \vartheta^n_j \triangle X_j\Big)^2 \Big| \mathcal{F}_k\Big] \\
&\geq \operatorname{ess\,inf}_{\vartheta' \in \mathcal{A}_k} E\Big[\Big(H - x - \sum_{j=1}^{k} \vartheta_j \triangle X_j - \sum_{j=k+1}^{T} \vartheta'_j \triangle X_j\Big)^2 \Big| \mathcal{F}_k\Big] \\
&= Y_k\Big(x + (\vartheta \cdot X)_k\Big),
\end{aligned}
$$

and so we have the submartingale property. The integrability then follows from

$$
Y_T\Big(x + (\vartheta \cdot X)_T\Big) = \Big(H - x - (\vartheta \cdot X)_T\Big)^2 \in L^1.
$$

(c) "⇒" Let $\vartheta^* \in \mathcal{A}$ be optimal. Clearly $(Y_k(x + (\vartheta^* \cdot X)_k))_{k=0,\dots,T}$ is adapted. To show that it is a martingale, we only need to show that

$$E[Y_T(c + (\vartheta^* \cdot X)_T)] = E[Y_0(c)].$$

By the optimality of ϑ^*, we have as in the lecture

$$\begin{aligned} E[Y_0(c)] &= E\Big[\operatorname{ess\,inf}_{\vartheta \in \mathcal{A}_0} E\Big[\Big(H - c - (\vartheta \cdot X)_T\Big)^2 \Big| \mathcal{F}_0\Big]\Big] \\ &= \inf_{\vartheta \in \mathcal{A}} E\Big[\Big(H - c - (\vartheta \cdot X)_T\Big)^2\Big] \\ &= E\Big[\Big(H - c - (\vartheta^* \cdot X)_T\Big)^2\Big] = E\Big[Y_T\Big(c + (\vartheta \cdot X)_T\Big)\Big]. \end{aligned}$$

This gives the desired equality.

"⇐" Suppose that $(Y_k(x + (\vartheta^* \cdot X)_k))_{k=0,\dots,T}$ is a martingale. Then using $Y_T(c + (\vartheta^* \cdot X)_T) = (H - c - (\vartheta^* \cdot X)_T)^2$ gives

$$\inf_{\vartheta \in \mathcal{A}} E\Big[\Big(H-c-(\vartheta^* \cdot X)_T\Big)^2\Big] = E[Y_0(c)] = E\Big[Y_T\Big(c+(\vartheta^* \cdot X)_T\Big)\Big] = E\Big[\Big(H-c-(\vartheta^* \cdot X)_T\Big)^2\Big],$$

which shows that ϑ^* is optimal.

(d) By part (b), we have for every fixed $\vartheta' \in \mathcal{A}_{k-1}$ that the process $Y_\cdot(x + (\vartheta' \cdot X)_\cdot)$ is a submartingale. Because $(\vartheta' \cdot X)_{k-1} = 0$ and $(\vartheta' \cdot X)_k = \vartheta'_k \Delta X_k$, this gives

$$Y_{k-1}(x) = Y_{k-1}\Big(x+(\vartheta' \cdot X)_{k-1}\Big) \le E\Big[Y_k\Big(x+(\vartheta' \cdot X)_k\Big)\Big|\mathcal{F}_{k-1}\Big] = E[Y_k(x+\vartheta'_k \triangle X_k)|\mathcal{F}_{k-1}].$$

Taking the ess inf yields

$$Y_{k-1}(x) \le \operatorname{ess\,inf}_{\vartheta' \in \mathcal{A}_{k-1}} E[Y_k(x + \vartheta'_k \triangle X_k)|\mathcal{F}_{k-1}].$$

To show "≥", we fix $\vartheta \in \mathcal{A}_{k-1}$ and then compute

$$\begin{aligned} E[Y_k(x + \vartheta_k \triangle X_k)|\mathcal{F}_{k-1}] &\le E\Bigg[E\Bigg[\Bigg(H - (x + \vartheta_k \triangle X_k) - \sum_{j=k+1}^{T} \vartheta_j \triangle X_j\Bigg)^2 \Bigg| \mathcal{F}_k\Bigg]\Bigg| \mathcal{F}_{k-1}\Bigg] \\ &= E\Bigg[\Bigg(H - x - \sum_{j=k}^{T} \vartheta_j \triangle X_j\Bigg)^2 \Bigg| \mathcal{F}_{k-1}\Bigg]. \end{aligned}$$

Taking the ess inf on both sides, we get

$$\begin{aligned} &\operatorname{ess\,inf}_{\vartheta \in \mathcal{A}_{k-1}} E[Y_k(x + \vartheta_k \triangle X_k)|\mathcal{F}_{k-1}] \\ &\qquad \le \operatorname{ess\,inf}_{\vartheta \in \mathcal{A}_{k-1}} E\Bigg[\Bigg(H - x - \sum_{j=k}^{T} \vartheta_j \triangle X_j\Bigg)^2 \Bigg| \mathcal{F}_{k-1}\Bigg] = Y_{k-1}(x). \end{aligned}$$

Exercise 10.3 Consider a general arbitrage-free single-period market with $\mathcal{F}_0$ trivial. Fix x and let $U : (0, \infty) \to \mathbb{R}$ be a concave, increasing, continuously differentiable (utility) function such that

$$\sup_{\vartheta \in \mathcal{A}(x)} E[U(x + \vartheta \cdot \triangle X_1)] < \infty, \tag{1}$$

with

$$\mathcal{A}(x) = \{\vartheta \in \mathbb{R}^d : x + \vartheta \cdot \triangle X_1 \geq 0 \ P\text{-a.s.}, U(x + \vartheta \cdot \Delta X_1) \in L^1\}.$$

Furthermore, assume that the supremum is attained in an interior point ϑ^* of $\mathcal{A}(x)$.

Show that we have the *first order condition*

$$E[U'(x + \vartheta^* \cdot \triangle X_1)\triangle X_1] = 0.$$

Hint: You may use that due to concavity,

$$y \mapsto \frac{U(y) - U(z)}{y - z}, \quad y \in (0, \infty) \setminus \{z\}$$

is nonincreasing. By optimality, ϑ^* is better than $\vartheta^* + \varepsilon\eta$ for any $\eta \neq 0$ and $0 < \varepsilon \ll 1$; so take the difference of the corresponding utilities, divide by ε and look at $\varepsilon \searrow 0$. Exploit the hint to see that this quantity is monotonic in ε.

Solution 10.3

Let η be any non-zero vector. Then by the assumption that ϑ^* is an interior point, $\vartheta^* + \varepsilon\eta \in \mathcal{A}(x)$ for all $0 < \varepsilon \ll 1$. Define

$$\triangle_\varepsilon^\eta = \frac{U(x + (\vartheta^* + \varepsilon\eta) \cdot \triangle X_1) - U(x + \vartheta^* \cdot \triangle X_1)}{\varepsilon},$$

for small ε as above. On $\{\eta \cdot \triangle X_1 = 0\}$, $\triangle_\varepsilon^\eta \equiv 0$, and on $\{\eta \cdot \triangle X_1 \neq 0\}$,

$$\triangle_\varepsilon^\eta = \eta \cdot \triangle X_1 \frac{U(x + (\vartheta^* + \varepsilon\eta) \cdot \triangle X_1) - U(x + \vartheta^* \cdot \triangle X_1)}{\varepsilon\eta \cdot \triangle X_1},$$

so $\triangle_\varepsilon^\eta$ is monotonically[1] increasing to $\eta \cdot \triangle X_1 U'(x + \vartheta^* \cdot \triangle X_1)$ as $\varepsilon \searrow 0$.

Note that all $\triangle_\varepsilon^\eta \in L^1(P)$, so that we can use monotone convergence. Moreover, by optimality, $E[\triangle_\varepsilon^\eta] \leq 0$ and therefore, by monotone convergence,

$$-\infty < E[\triangle_\varepsilon^\eta] \leq E[U'(x + \vartheta^* \cdot \triangle X_1)\eta \cdot \triangle X_1] = \lim_{\varepsilon \searrow 0} E[\triangle_\varepsilon^\eta] \leq 0.$$

Replacing η by $-\eta$ gives also ≥ 0; so $E[U'(x + \vartheta^* \cdot \triangle X_1)\eta \cdot \triangle X_1] = 0$. Finally, since η can be chosen arbitrary, we can take $\eta = e^i$ for $i = 1, ..., d$ to get

$$E[U'(x + \vartheta^* \cdot \triangle X_1)\triangle X_1] = 0.$$

[1] This is easily seen by splitting into two cases depending on the sign of $\eta \cdot \triangle X_1$.