Introduction to Mathematical Finance

Exercise sheet 11

Please submit your solutions online until Wednesday 10pm, 15/05/2023.

Exercise 11.1 Recall that $\mathcal{C}(x) := \{f \in L^0_+ : f \leq V_T \text{ for some } V \in \mathcal{V}(x)\}$ and $\mathcal{D}(z) := \{h \in L^0_+ : h \leq Z_T \text{ for some } Z \in \mathcal{Z}(z)\}$.

(a) Show that $\mathcal{C}(x)$ and $\mathcal{D}(z)$ are both convex and solid (i.e., $Y \in A$ and $Y' \leq Y$ implies $Y' \in A$).

(b) Show that $j(z) := \inf_{Z \in \mathcal{Z}(z)} E[J(Z_T)] = \inf_{h \in \mathcal{D}(z)} E[J(h)]$.

(c) Show that $E[J(Z_T)]$, for $Z \in \mathcal{Z}(z)$, is always well defined in $(-\infty, +\infty]$.

Exercise 11.2 Let $U : \mathbb{R} \to \mathbb{R}$ be a strictly increasing utility function and consider a general arbitrage-free market in finite discrete time, with horizon $T \in \mathbb{N}$ and with $\mathcal{F}_0$ trivial. Recall that $\mathcal{C} = G_T(\Theta) - L^0_+$.

(a) Show that an optimizer for

$$u(x) = \sup_{\vartheta \in \Theta} E\left[U(x + G_T(\vartheta)\right]$$

can be obtained from an optimizer for

$$u_{\mathcal{C}}(x) = \sup_{f \in \mathcal{C}} E\left[U(x + f)\right],$$

and vice versa.

(b) Denote by $\mathbb{P}_a$ the set of absolutely continuous martingale measures. Show that if Ω is finite and $f \in L^0$, then

$$f \in \mathcal{C} \quad \Longleftrightarrow \quad E_Q[f] \leq 0, \quad \forall Q \in \mathbb{P}_a.$$

Exercise 11.3 Consider a general market in finite discrete time with horizon $T \in \mathbb{N}$. Let $U : (0, \infty) \to \mathbb{R}$ be an increasing and concave utility function, and denote by u the indirect utility from maximizing the utility of final wealth, i.e.,

$$u(x) = \sup_{\theta \in \Theta^x_{adm}} E\Big[U\Big(x + G_T(\vartheta)\Big)\Big],$$

for $x > 0$, where $\Theta^x_{adm} = \{\vartheta \in \Theta : \vartheta \text{ is } x\text{-admissible}\}$.

(a) Assume that $u(x_0) < \infty$ for some $x_0 > 0$. Show that u is increasing, concave and $u(x) < \infty$ for all $x > 0$.

(b) Show that if U is unbounded from above and the market admits an arbitrage opportunity, then $u \equiv +\infty$. What happens if U is not unbounded from above?

Exercise 11.4

(a) Suppose that $U : (0, \infty) \mapsto \mathbb{R}$ is strictly increasing, strictly concave and C^1. Show that for any $Q \in \mathbb{P}_e$, we have

$$\sup_{f \in L^0} E\left[U(f) - f\lambda \frac{dQ}{dP}\right] = E\left[\sup_{z>0} \left(U(z) - z\lambda \frac{dQ}{dP}\right)\right].$$

(b) Using the notations from Theorem IV.0.5 and Theorem IV.0.3, show that $Q^* = Q^*(\lambda^*)$, i.e., the measure Q^* constructed in the proof of Theorem IV.0.5 coincides with the optimal $Q^*(\lambda^*)$ for the dual problem in Theorem IV.0.3 with the parameter $\lambda = \lambda^*$ from the proof of Theorem IV.0.5.