Introduction to Mathematical Finance

Exercise sheet 11

Please submit your solutions online until Wednesday 10pm, 15/05/2023.

Exercise 11.1 Recall that $\mathcal{C}(x) := \{f \in L^0_+ : f \leq V_T \text{ for some } V \in \mathcal{V}(x)\}$ and $\mathcal{D}(z) := \{h \in L^0_+ : h \leq Z_T \text{ for some } Z \in \mathcal{Z}(z)\}$.

(a) Show that $\mathcal{C}(x)$ and $\mathcal{D}(z)$ are both convex and solid (i.e., $Y \in A$ and $Y' \leq Y$ implies $Y' \in A$).

(b) Show that $j(z) := \inf_{Z \in \mathcal{Z}(z)} E[J(Z_T)] = \inf_{h \in \mathcal{D}(z)} E[J(h)]$.

(c) Show that $E[J(Z_T)]$, for $Z \in \mathcal{Z}(z)$, is always well defined in $(-\infty, +\infty]$.

Solution 11.1

(a) The fact that $\mathcal{C}(x)$ and $\mathcal{D}(z)$ are solid is direct from their definitions. We show that $\mathcal{D}(z)$ is convex. The argument for $\mathcal{C}(x)$ is analogous. Let $h_1, h_2 \in \mathcal{D}(z)$ with $h_1 \leq Z^1_T$ and $h_2 \leq Z^2_T$ and $Z^1, Z^2 \in \mathcal{Z}(z)$. Then for $\lambda \in (0,1)$, we have

$$\lambda h_1 + (1-\lambda)h_2 \leq \lambda Z^1_T + (1-\lambda)Z^2_T.$$

Moreover, the process $\lambda Z^1 + (1-\lambda)Z^2$ is still a nonnegative adapted process with $\lambda Z^1_0 + (1-\lambda)Z^2_0 = z$ and $(\lambda Z^1 + (1-\lambda)Z^2)V$ being a supermartingale for all $V \in \mathcal{V}(1)$. Hence $\lambda Z^1 + (1-\lambda)Z^2 \in \mathcal{Z}(z)$ and $\mathcal{D}(z)$ is convex.

(b) For "$\geq$", we just notice that $\mathcal{Z}_T(z) \subseteq \mathcal{D}(z)$. For "$\leq$", we use that J is decreasing and $h \leq Z_T$ to obtain $J(Z_T) \leq J(h)$ which implies $E[J(Z_T)] \leq E[J(h)]$. Taking suprema on both sides yields the conclusion.

(c) For any $x > 0$, $J(Z_T) \geq U(x) - xZ_T$ gives $E[Z_T] \geq U(x) - xE[Z_T]$ and $E[Z_T] \leq z$; so

$$E[J(Z_T)] \geq \sup_{x>0} \Big(U(x) - xz \Big) = J(z) > -\infty.$$

Exercise 11.2 Let $U : \mathbb{R} \to \mathbb{R}$ be a strictly increasing utility function and consider a general arbitrage-free market in finite discrete time, with horizon $T \in \mathbb{N}$ and with $\mathcal{F}_0$ trivial. Recall that $\mathcal{C} = G_T(\Theta) - L^0_+$.

(a) Show that an optimizer for

$$u(x) = \sup_{\vartheta \in \Theta} E\left[U(x + G_T(\vartheta)\right]$$

can be obtained from an optimizer for

$$u_{\mathcal{C}}(x) = \sup_{f \in \mathcal{C}} E\left[U(x + f)\right],$$

and vice versa.

(b) Denote by $\mathbb{P}_a$ the set of absolutely continuous martingale measures. Show that if Ω is finite and $f \in L^0$, then

$$f \in \mathcal{C} \iff E_Q[f] \le 0, \quad \forall Q \in \mathbb{P}_a.$$

Solution 11.2

(a) First note that $G_T(\Theta) \subseteq \mathcal{C}$. Therefore, $u_{\mathcal{C}}(x) \ge u(x)$.

Suppose f^* is a maximizer. Then, since $f^* \in \mathcal{C}$, $f^* = G_T(\vartheta^*) - Y$ for some $\vartheta^* \in \Theta$ and $Y \ge 0$, and

$$u_{\mathcal{C}}(x) = E\Big[U\Big(x + G_T(\vartheta^*) - Y\Big)\Big] \le E\Big[U\Big(x + G_T(\vartheta^*)\Big)\Big] \le u(x).$$

Since U is strictly increasing, Y must be identically zero because otherwise the first inequality above becomes strict. Hence, $u_{\mathcal{C}}(x) = u(x)$, and the optimizer f^* corresponds to an optimizer ϑ^* for the first problem.

On the other hand, if ϑ^* is an optimizer of the first problem, then $f^* = G_T(\vartheta^*)$ must optimize the second, for otherwise there would exist a strictly better f', and by the argument above also a strictly better ϑ', violating the assumption that ϑ^* is an optimizer.

(b) Since Ω is finite, every f is bounded from below by $\min_\omega f$. Therefore, by Theorem II.7.2,

$$f \in \mathcal{C} \iff E_Q[f] \le 0, \quad \forall Q \in \mathbb{P}_e.$$

We need to extend this statement to $\mathbb{P}_a$. If $E_Q[f] \le 0$ for all $Q \in \mathbb{P}_a$, the desired implication holds trivially. On the other hand, suppose $f \in \mathcal{C}$. Then $E_Q[f] \le 0$ for all EMMs Q. Thus,

$$\sup_{Q \in \mathbb{P}_e} E_Q[f] \le 0,$$

and, by Exercise 3.1,

$$\sup_{Q \in \mathbb{P}_a} E_Q[f] \le 0.$$

This is what we wanted to show.

Exercise 11.3 Consider a general market in finite discrete time with horizon $T \in \mathbb{N}$. Let $U : (0, \infty) \to \mathbb{R}$ be an increasing and concave utility function, and denote by u the indirect utility from maximizing the utility of final wealth, i.e.,

$$u(x) = \sup_{\theta \in \Theta^x_{adm}} E\Big[U\Big(x + G_T(\vartheta)\Big)\Big],$$

for $x > 0$, where $\Theta^x_{adm} = \{\vartheta \in \Theta : \vartheta \text{ is } x\text{-admissible}\}$.

(a) Assume that $u(x_0) < \infty$ for some $x_0 > 0$. Show that u is increasing, concave and $u(x) < \infty$ for all $x > 0$.

(b) Show that if U is unbounded from above and the market admits an arbitrage opportunity, then $u \equiv +\infty$. What happens if U is not unbounded from above?

Solution 11.3

(a) For any $x \leq y$, we have that

$$E\Big[U\Big(x + G_T(\vartheta)\Big)\Big] \leq E\Big[U\Big(y + G_T(\vartheta)\Big)\Big].$$

Taking the supremum on both sides yields $u(x) \leq u(y)$.

Let $z = \lambda x + (1 - \lambda)y$ for some $\lambda \in [0, 1]$. For any $\vartheta_x \in \Theta^x_{adm}$ and $\vartheta_y \in \Theta^y_{adm}$, it follows from linearity of $G_T(\cdot)$ that

$$z + G_T\Big(\lambda\vartheta_x + (1 - \lambda)\vartheta_y\Big) = \lambda\Big(x + G_T(\vartheta_x)\Big) + (1 - \lambda)\Big(y + G_T(\vartheta_y)\Big) \geq 0,$$

i.e., $\vartheta_z := \lambda\vartheta_x + (1 - \lambda)\vartheta_y \in \Theta^z_{adm}$. Finally, using the above inequality and the concavity of U,

$$E\Big[U\Big(z + G_T(\vartheta_z)\Big)\Big] \geq \lambda E\Big[U\Big(x + G_T(\vartheta_x)\Big)\Big] + (1 - \lambda)E\Big[U\Big(y + G_T(\vartheta_y)\Big)\Big].$$

Taking the supremum over ϑ_x and ϑ_y preserves the inequality, showing that u is also concave.

Let x be any point. By monotonicity, we are done if $x \leq x_0$, so assume that $x > x_0$. Let $y \in (0, x_0)$. Then $x_0 = \lambda x + (1 - \lambda)y$ for some $\lambda \in (0, 1)$. By concavity,

$$u(x_0) \geq \lambda u(x) + (1 - \lambda)u(y),$$

showing that $u(x)$ is finite.

(b) Let ϑ^a denote an arbitrage opportunity with $G_T(\vartheta^a) \geq 0$ P-a.s. and $G_T(\vartheta^a) > 0$ on some set A with $P[A] > 0$. Hence, $\vartheta^a \in \Theta^x_{adm}$ for every x, and the same holds for $n\vartheta^a$, $n \in \mathbb{N}$. Thus,

$$u(x) \geq E[U(x)1_{A^c}] + E\Big[U\Big(x + nG_T(\vartheta^a)\Big)1_A\Big].$$

By monotone convergence, the second term converges to $E[U(\infty)1_A]$ as $n \to \infty$, and by the assumption that U is unbounded, this value is infinite. Thus, $u(x) = +\infty$ for every x.

Suppose that U is bounded from above. So $U(\infty) := \lim x \to \infty U(x)$ exists in $\mathbb{R}$. Set $A := \{G_T(\vartheta^a) > 0\}$ and $P[A] = \alpha > 0$. Then $(x + nG_T(\vartheta^a))\mathbb{1}_A \to \infty \mathbb{1}_A$ which implies $u(x) \geq (1 - \alpha)U(x) + \alpha U(\infty)$. This in turn yields

$$\liminf_{x \to \infty} \frac{u(x)}{U(x)} \geq 1.$$

But clearly $u(x) \leq U(\infty)$ for all x. So we have

$$\limsup_{x \to \infty} \frac{u(x)}{U(x)} \leq 1,$$

and therefore

$$\lim_{x \to \infty} \frac{u(x)}{U(x)} = 1.$$

Exercise 11.4

(a) Suppose that $U : (0, \infty) \mapsto \mathbb{R}$ is strictly increasing, strictly concave and C^1. Show that for any $Q \in \mathbb{P}_e$, we have

$$\sup_{f \in L^0} E\left[U(f) - f\lambda \frac{dQ}{dP}\right] = E\left[\sup_{z>0}\left(U(z) - z\lambda \frac{dQ}{dP}\right)\right].$$

(b) Using the notations from Theorem IV.0.5 and Theorem IV.0.3, show that $Q^* = Q^*(\lambda^*)$, i.e., the measure Q^* constructed in the proof of Theorem IV.0.5 coincides with the optimal $Q^*(\lambda^*)$ for the dual problem in Theorem IV.0.3 with the parameter $\lambda = \lambda^*$ from the proof of Theorem IV.0.5.

Solution 11.4

(a) " $\leq$ " is clear. For " $\geq$ ", note that $\sup_{z>0}\Big(U(z) - zy\Big) = J(y)$ for $y > 0$ is attained in $z = (U')^{-1}(y)$. So if we set $\tilde{f} := (U')^{-1}(\lambda \frac{dQ}{dP})$, then $\tilde{f} \in L^0$ and

$$E\left[\sup_{z>0}\left(U(z) - z\lambda \frac{dQ}{dP}\right)\right] = E\left[U(\tilde{f}) - \tilde{f}\lambda \frac{dQ}{dP}\right] \leq \sup_{f \in L^0} E\left[U(f) - f\lambda \frac{dQ}{dP}\right].$$

(b) Using the notations from the lectures,

$$E\left[J\left(\lambda^* \frac{dQ^*}{dP}\right)\right] = E\big[U(f^*)\big] - \lambda^* x \leq E\left[J\left(\lambda^* \frac{dQ}{dP}\right)\right] \quad \forall Q,$$

hence $Q^* = Q^*(\lambda^*)$.