# Introduction to Mathematical Finance

## Exercise sheet 12

**Exercise 12.1**

(a) Prove the uniqueness of the solution $h_z^*$ to the dual problem.

(b) Assuming $z \neq z'$ and $j(z), j(z') < \infty$, prove that $P[h_z^* \neq h_{z'}^*] > 0$.

**Solution 12.1**

(a) Suppose to the contrary that $h_z^*, \tilde{h}_z^*$ are two solutions with $P[h_z^* \neq \tilde{h}_z^*] > 0$. Set $h_0 := \frac{1}{2}(h_z^* + \tilde{h}_z^*)$. Since $J$ is strictly convex, we have on $\{h_z^* \neq \tilde{h}_z^*\}$ $J(h_0) < \frac{1}{2}(J(h_z^*) + J(\tilde{h}_z^*))$, and $J(h_0) \leq 1/2\Big(J(h_z^*) + J(\tilde{h}_z^*)\Big)$ $P-$a.s. Because $\{h_z^* \neq \tilde{h}_z^*\}$ has positive probability, we obtain

$$E[J(h_0)] < \frac{1}{2}E[(J(h_z^*) + J(\tilde{h}_z^*))] = j(z).$$

But note that $h_0 \in \mathcal{D}(z)$ due to the convexity of $\mathcal{D}(z)$. This contradicts the optimality of $h_z^*$.

(b) Suppose to the contrary that $h_z^* = h_{z'}^*$ $P$-a.s. for $z < z'$. Then $j(z') = E[J(h_{z'}^*)] = E[J(h_z^*)]$. Since $h_z^* \leq Z_T$ for some $Z \in \mathcal{Z}(z)$, then the process $Z' := Z + (z' - z)Z \in \mathcal{Z}(z')$ and $h_z^* + (z' - z)Z_T \leq Z'_T$, hence $h_z^* + (z' - z)Z_T \in \mathcal{D}(z')$. Since $J$ is strictly decreasing, we have $E[J(h_z^* + (z' - z)Z_T)] < E[J(h_z^*)] = j(z')$, which contradicts the optimality of $h_{z'}^*$ because $h_z^* + (z' - z)Z_T \neq h_{z'}^*$ $P$-a.s.

**Exercise 12.2**

(a) Analogously to the proof of Lemma IV.5.2 show that, for fixed $0 < \mu < 1$ we can find a constant $\tilde{C} < \infty$ and $y_0 > 0$ such that

$$-J'(\mu y) < \tilde{C}\frac{J(y)}{y} \quad \text{for } 0 < y < y_0.$$

(b) Prove that if $z_n \to z$ and all $z_n$ and $z$ are in the interior of $\{j < \infty\}$ and $\mu_n \uparrow 1$, then

$$\lim_{n\to\infty} E[h^*_{z_n} I(\mu_n h^*_{z_n})] = E[h^*_z I(h^*_z)].$$

*Hint:* Use (a) and almost repeat the proof of Lemma IV.5.3.

**Solution 12.2**

(a) From Lemma IV.5.2 we know that we can find a constant $C < \infty$ and $y_0 > 0$ such that

$$-J'(y) < C\frac{J(y)}{y} \quad \text{for } 0 < y < y_0,$$

and hence for $\hat{C} = C/\mu$ we get

$$-J'(\mu y) < \hat{C}\frac{J(\mu y)}{y}.$$

Since $J$ is convex and $\mu < 1$, then $J(y) \geq J(\mu y) + J'(\mu y)(y - \mu y)$, that means that $J(\mu y) \leq J(y) - J'(\mu y)y(1-\mu)$, so that

$$-J'(\mu y) < \hat{C}\biggl(\frac{J(y)}{y} - J'(\mu y)(1-\mu)\biggr),$$

and hence after defying $\tilde{C} = \hat{C}/\mu$ we get

$$-J'(\mu y) < \tilde{C}\frac{J(y)}{y} \quad \text{for } 0 < y < y_0.$$

(b) We first rewrite $h^*_{z_n} I(\mu_n h^*_{z_n}) = \mu_n^{-1}(\mu_n h^*_{z_n} I(\mu_n h^*_{z_n}))$; so it suffices to show that

$$\lim_{n\to\infty} E[\mu_n h^*_{z_n} I(\mu_n h^*_{z_n})] = E[h^*_z I(h^*_z)].$$

We now argue as in the proof of Lemma IV.5.3.

First by Lemma IV.5.1 and continuity of $I \geq 0$, $\mu_n h^*_{z_n} I(\mu_n h^*_{z_n}) \to h^*_z I(h^*_z)$ in $L^0$. So we only need to prove uniform integrability.

**I.** This part is the same. Since $I$ is decreasing and $U$ is increasing, we get for $y \geq y_0 \geq 0$ that

$$0 \leq yI(y) = U\bigl(I(y)\bigr) - J(y) \leq U\bigl(I(y_0)\bigr) + J^-(y)$$

and therefore

$$0 \leq X_n := \mu_n h^*_{z_n} I(\mu_n h^*_{z_n}) \mathbb{1}_{\{\mu_n h^*_{z_n} \geq y_0\}} \leq \Big| U\Big( I(y_0) \Big) \Big| + J^-(\mu_n h^*_{z_n}).$$

If $z_n \to z$, $\mu_n \to \mu$, then $(z_n)$ is bounded by some $z'$ and $(\mu_n)$ is bounded by some $\mu'$, say, and so all the $\mu_n h^*_{z_n}$ lie in $\mathcal{D}(\mu' z')$. But we know from IV.3.3 that the family $\{J^-(h) : h \in \mathcal{D}(\mu' z')\}$ is uniformly integrable, and so also $(X_n)_{n \in \mathbb{N}}$ is uniformly integrable.

**II.** From (a) it follows that there exist $C < \infty$ and $y_0 > 0$ such that

$$0 \leq \mu_n h^*_{z_n} I(\mu_n h^*_{z_n}) \mathbb{1}_{\{\mu_n h^*_{z_n} < y_0\}} \leq C |J(h^*_{z_n})|,$$

so it is enough to prove that $(|J(h^*_{z_n})|)_{n \in \mathbb{N}}$ is uniformly integrable, but the fact that such a sequence is uniformly integrable is shown in the proof of Lemma IV.5.3.

**Exercise 12.3** Consider a general market in finite discrete time with horizon $T \in \mathbb{N}$. Let $U : (0, \infty) \to \mathbb{R}$ be an increasing and concave utility function, and denote by $u$ the indirect utility from maximizing the utility of final wealth, i.e.,

$$u(x) = \sup_{\vartheta \in \Theta^x} E\Big[U\Big(x + G_T(\vartheta)\Big)\Big],$$

for $x > 0$, where $\Theta^x = \{\vartheta \in \Theta : \vartheta \text{ is } x\text{-admissible}\}$.

Suppose that $U$ is strictly increasing, $U(\infty) < \infty$ and $X$ satisfies NA. Show that if there exists an optimal strategy $\vartheta^*$ for $x$, then $u(x) < U(\infty)$.

**Solution 12.3** Fix $x > 0$. Let $\vartheta^*$ be an optimal strategy. Denote $A := \{x + G_T(\vartheta^*) = \infty\}$. Since $X$ satisfies NA, then $P(A) < 1$. Then, since $U$ is strictly increasing:

$$u(x) = P[A]U(\infty) + E\Big[I(A^c)U\Big(x + G_T(\vartheta^*)\Big)\Big] < U(\infty).$$