Introduction to Mathematical Finance

Exercise sheet 1

Please submit your solutions online until Wednesday 22:00, 28/02/2024.

Exercise 1.1 Let $\mathcal{C} := \mathbb{R} \times \mathbb{R}^K$ be the consumption space, $\mathcal{D}$ the payoff matrix, e^i an endowment and π a price vector. Recall the budget set

$$B(e^i, \pi) := \{c \in \mathcal{C} : \exists \vartheta \in \mathbb{R}^N \text{ with } c_0 \leq e_0^i - \vartheta \cdot \pi \text{ and } c_T \leq e_T^i + \mathcal{D}\vartheta\}.$$

(a) For the statements

1. $c \in B(e^i, \pi)$,
2. $c - e^i \in B(0, \pi)$,
3. $c - e^i$ is attainable from 0 initial wealth endowment,

show that (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3).

(b) Show by an example that (2) $\Rightarrow$ (3) is not true in general.

Solution 1.1

(a) By definition, $c \in B(e^i, \pi)$ if and only if there exists $\vartheta \in \mathbb{R}^N$ with $c_0 \leq e_0^i - \vartheta \cdot \pi$ and $c_T \leq e_T^i + \mathcal{D}\vartheta$, or more compactly $c \leq e^i + \bar{\mathcal{D}}\vartheta$. This is equivalent to $c - e^i \leq \bar{\mathcal{D}}\vartheta$ and hence to $c - e^i \in B(0, \pi)$.

If $c - e^i$ is attainable with 0 initial wealth endowment, there exists $\hat{\vartheta} \in \mathbb{R}^N$ such that $c_0 - e_0^i = -\hat{\vartheta} \cdot \pi$ and $c_T - e_T^i = \mathcal{D}\hat{\vartheta}$ (or more compactly $c - e^i = \bar{\mathcal{D}}\hat{\vartheta}$), which shows that $c - e^i \in B(0, \pi)$.

(b) The idea is simply to find a nonattainable consumption which still lies in the budget set. To do this, we consider a payoff matrix without full rank because otherwise every consumption is attainable. Let

$$\pi := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathcal{D} := \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Clearly $\mathcal{D}(\mathbb{R}^2) = \{(a, 2a)^{\text{tr}} : a \in \mathbb{R}\}$. Take for instance $c_0 = e_0^i - 1$, $\vartheta = (1, 0)^{\text{tr}}$, $c_T = e_T^i + (1, 1.5)^{\text{tr}}$. Then

$$c_0 - e_0^i = -1 \leq -(1, 0) \cdot (1, 1),$$
$$c_T - e_T^i = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus, $c - e^i \in B(0, \pi)$. But clearly $(1, 1.5)^{\text{tr}} \notin \mathcal{D}(\mathbb{R}^2)$, which shows that $c - e^i$ is not attainable with 0 initial wealth endowment.

Exercise 1.2 Let $\succeq$ be a preference order on $\mathcal{C}$ satisfying axioms (P1)-(P5). A function $\mathcal{U} : \mathcal{C} \to \mathbb{R}$ is called a *utility functional representing* $\succeq$ or *a numerical representation of* $\succeq$ if

$$c' \succeq c \quad \Longleftrightarrow \quad \mathcal{U}(c') \geq \mathcal{U}(c).$$

(a) Show that all $\mathcal{U}$ representing $\succeq$ must be *quasiconcave*, i.e., for all $c, c' \in \mathcal{C}$ and $\lambda \in [0, 1]$,

$$\mathcal{U}\Big(\lambda c + (1 - \lambda)c'\Big) \geq \min\{\mathcal{U}(c), \mathcal{U}(c')\}.$$

(b) Which axioms are needed for this result?

(c) Show by a counterexample that a preference order can be represented by a utility functional which is not concave.

Solution 1.2

(a) Let c' and c be arbitrary elements of $\mathcal{C}$. Without loss of generality, assume that $c' \succeq c$. Then by convexity, $\lambda c' + (1 - \lambda)c \succeq c$, and hence

$$\mathcal{U}\Big(\lambda c' + (1 - \lambda)c\Big) \geq \mathcal{U}(c) = \min\{\mathcal{U}(c), \mathcal{U}(c')\}.$$

(b) In the solution above, we implicitly used completeness to assume $c' \succsim c$, and we used convexity and that $c \succeq c$.

(c) Set $\mathbf{1} := (1, ..., 1)^{\text{tr}} \in \mathbb{R} \times \mathbb{R}^N$ and define

$$c' \succeq c \quad :\Longleftrightarrow \quad c' \cdot \mathbf{1} \geq c \cdot \mathbf{1}.$$

It is easy to check that this satisfies the axioms (P1)–(P5). The natural utility functional is then given by

$$\mathcal{U}(c) = c \cdot \mathbf{1}.$$

However, since $\exp(\cdot)$ is increasing, it will preserve the order. Hence $\exp(\mathcal{U}(\cdot))$ is also a utility functional which represents $\succeq$, but it is not concave. More generally, exp can be replaced by any strictly increasing and non-concave function on $\mathbb{R}$.

Exercise 1.3

(a) Construct a market with arbitrage of the first kind but with no arbitrage of the second kind.

(b) Construct a market with arbitrage of the second kind but with no arbitrage of the first kind.

(c) Find a sufficient condition under which existence of an arbitrage of the second kind implies the existence of an arbitrage of the first kind.

Solution 1.3

(a) Consider a market consisting of a single asset with $\pi = 0$, $\mathcal{D} = (1, 2)^{\text{tr}}$. Set $\vartheta = 1$. Clearly, $\mathcal{D}\vartheta = (1, 2)^{\text{tr}} \geq 0$ and $\mathcal{D}\vartheta(\{\omega_i\}) > 0$ for both $i = 1, 2$. Thus ϑ is an arbitrage opportunity of the first kind. However, since $\pi = 0$, there exists no arbitrage of the second kind.

(b) Consider the situation where $\pi = 1$ and $\mathcal{D} = (0, 0)$. Then $\vartheta < 0$ would be an arbitrage of the second kind. But since $\mathcal{D}$ vanishes, we have for any $\tilde{\vartheta} \in \mathbb{R}$ that $\mathcal{D}\tilde{\vartheta} = (0, 0)^{\text{tr}}$. So there exists no arbitrage of the first kind.

(c) Suppose for instance there is an asset $D^\ell \geq 0$ and $D^\ell \not\equiv 0$ and $\pi^\ell > 0$. Let ϑ be an arbitrage opportunity of the second kind. Set $\alpha = -\vartheta \cdot \pi / \pi^\ell > 0$. We consider a new strategy $\hat{\vartheta} = \vartheta + \alpha e_\ell$ where e_ℓ is the vector with 1 in its ℓth component and 0 elsewhere. Then $\hat{\vartheta} \cdot \pi = \vartheta \cdot \pi + \alpha \cdot \pi^\ell = 0$ and $\mathcal{D}\hat{\vartheta} = \mathcal{D}\vartheta + \alpha \mathcal{D}^\ell \geq 0$. Since $\mathcal{D}\vartheta \geq 0$ and $\alpha \mathcal{D}^\ell \geq 0$ with $\alpha \mathcal{D}^\ell \not\equiv 0$, we have $\mathcal{D}\hat{\vartheta} \geq 0$ and $\mathcal{D}\hat{\vartheta} \not\equiv 0$. Hence, $\hat{\vartheta}$ is an arbitrage opportunity of the first kind.[1]

[1] This is just part of Proposition I.3.1.