Introduction to Mathematical Finance

Exercise sheet 2

Please submit your solutions online until Wednesday 22:00, 06/03/2024.

Exercise 2.1 Denote by $\mathcal{P}$ the collection of all EMMs and denote by $|\mathcal{P}|$ the cardinality of $\mathcal{P}$. Prove or disprove:

$$\text{Completeness} \iff |\mathcal{P}| \leq 1.$$

Solution 2.1 "$\Longrightarrow$" Suppose the market is complete. Then either the market is free of arbitrage or not. In the first case $|\mathcal{P}| = 1$. In the second case $|\mathcal{P}| = 0$.

"$\Longleftarrow$" This is not true in general: if $|\mathcal{P}| = 1$, then the market is arbitrage-free and complete by Theorems I.4.3 and I.4.5.

If $|\mathcal{P}| = 0$, then the market has arbitrage. If we choose $\mathcal{D}$ such that $\text{rank}(\mathcal{D}) < K$, then the market is incomplete.

Exercise 2.2 Let D^0 be a numéraire, and define

$$\Psi^Q(c) = c_0 + E_Q\left[\frac{c_T}{D^0}\right] \quad \text{for } c \in \mathcal{C},$$
$$Q^\Psi[A] = \Psi(0, D^0 \mathbb{1}_A) \quad \text{for } A \in \mathcal{F}.$$

Here Q is a probability measure on $\mathcal{F}$, and Ψ is a linear functional in $\mathcal{C}$. Denote by J the mapping : $Q \mapsto J(Q) := \Psi^Q$. If Ψ is a consistent price system, check that $J(\hat{Q}) = \hat{\Psi}$.

Solution 2.2 In the following calculation we use first the definitions of J and Q^Ψ, and finally the linearity of Ψ to compute:

$$\begin{aligned} J(Q^\Psi)(c) &= c_0 + \sum_{k=1}^K Q^\Psi[\{\omega_k\}] \frac{c_T(\omega_k)}{D^0(\omega_k)} \\ &= c_0 + \sum_{k=1}^K \Psi\Big(0, D^0(\omega_k)\mathbb{1}_{\{\omega_k\}}\Big) \frac{c_T(\omega_k)}{D^0(\omega_k)} \\ &= c_0 \Psi(1, \underline{0}) + \sum_{k=1}^K \Psi\Big(0, c_T(\omega_k)\mathbb{1}_{\{\omega_k\}}\Big) = \Psi(c). \end{aligned}$$

Exercise 2.3 Consider the one-step *binomial market* defined by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+d \end{pmatrix}$$

for some $r > -1$ and $u > d$.

(a) Show that this market is free of arbitrage if and only if $u > r > d$,

(b) Construct an arbitrage opportunity for a market where $u = r > d$.

Solution 2.3

(a) The market is arbitrage-free if and only if there exists a probability measure $Q = (q_u, q_d)$ such that

$$1 = E_Q\left[\frac{D^1}{D^0}\right] = q_u\frac{1+u}{1+r} + q_d\frac{1+d}{1+r},$$

where q_u and q_d are positive and $q_u + q_d = 1$. These equalities are equivalent to

$$q_u = \frac{r-d}{u-d} \quad \text{and} \quad q_d = \frac{u-r}{u-d}.$$

Both q_u and q_d are positive if and only if $r \in (d, u)$.

(b) If $u = r$, the risky asset can only lose value relative to the risk-free asset. An arbitrage of the first kind is therefore given by ("go long risk-free asset and short risky asset")

$$\vartheta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This strategy costs $\vartheta \cdot \pi = 0$ at time 0 and yields

$$\mathcal{D}\vartheta = \begin{pmatrix} r-u \\ r-d \end{pmatrix} = \begin{pmatrix} 0 \\ r-d \end{pmatrix}$$

at time T.