Coordinator
Qinxin Yan

Introduction to Mathematical Finance

Exercise sheet 3

Please submit your solutions online until Wednesday 22:00, 13/03/2024.

Exercise 3.1 For a market $(\mathcal{D}, \pi)$ with a numéraire D^0, a *martingale measure* for numéraire D^0 is a probability measure $\mathbb{Q}$ on $\mathcal{F}$ with $E_{\mathbb{Q}}[\frac{D^l}{D^0}] = \pi^l$ for $l = 0, 1, ..., N$. We call $\mathbb{Q}$ *equivalent* to $\mathbb{P}$ if $\mathbb{Q}[\{\omega_k\}] > 0$ for $k = 1, ..., K$, and absolutely continuous with respect to P if $Q[\{\omega_k\}] \geq 0$ for $k = 1, ..., K$. Denote by $\mathcal{P}$ (resp. $\mathcal{P}_a$) the set of all equivalent (resp. absolutely continuous) martingale measures for the numéraire D^0. Consider an arbitrage-free market with numéraire D^0.

(a) Show that $\mathcal{P}_a = \overline{\mathcal{P}}$. Here we identity $\mathcal{P}$ with a subset of $\mathbb{R}^N_+$ and denote by $\overline{}$ the closure in $\mathbb{R}^N$.

(b) Use (a) to show that for any random variable X,

$$\sup_{\mathbb{Q} \in \mathcal{P}} E_{\mathbb{Q}}[X] = \sup_{\mathbb{Q} \in \mathcal{P}_a} E_{\mathbb{Q}}[X].$$

(c) Show that for any payoff H, the supremum

$$\sup_{\mathbb{Q} \in \mathcal{P}_a} E_{\mathbb{Q}}\left[\frac{H}{D^0}\right]$$

is attained in some $\mathbb{Q} \in \mathcal{P}_a$. Does this imply that the market is complete?

Exercise 3.2 Let

$$\pi = \begin{pmatrix} 1 \\ 1100 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1.1 & 1320 \\ 1.1 & 1210 \\ 1.1 & 880 \end{pmatrix}.$$

Denote by H the payoff of a put option with strike $K = 900$, i.e.

$$H = (900 - D^1)^+ = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix}.$$

(a) Find

$$\sup_{\mathbb{Q} \in \mathcal{P}} E_{\mathbb{Q}}\left[\frac{H}{D^0}\right].$$

(b) Compute

$$\inf\{\pi \cdot \vartheta : \vartheta \text{ with } \mathcal{D}\vartheta \geq H\}.$$

(c) Construct a market with $\mathcal{P}_a \neq \overline{\mathcal{P}}$.

Exercise 3.3 Consider the one-step *trinomial model* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$ and u, m, d with $u > m > d$ and $u > r > d$.

(a) Show that $\mathcal{P} \neq \emptyset$.

(b) Describe the set $\mathcal{P}$.

Hint: Use the $\mathbb{Q}-$probability of the 'middle outcome' as a parameter in a parametrization of $\mathcal{P}$ as a line segment in $\mathbb{R}^3$.

(c) Denote by $\mathcal{P}_a$ the set of all martingale measures $\mathbb{Q}$ which are absolutely continuous with respect to $\mathbb{P}$. An element $\mathbb{R} \in \mathcal{P}_a$ is an *extreme point* if $\mathbb{R}$ cannot be written as a strict convex combination of elements in $\mathcal{P}_a$, i.e. the condition $\mathbb{R} = \lambda\mathbb{Q} + (1-\lambda)\mathbb{Q}'$ with $0 < \lambda < 1$ and both $\mathbb{Q}, \mathbb{Q}' \in \mathcal{P}_a$ implies that $\mathbb{Q} = \mathbb{Q}'$. Find the extreme points of $\mathcal{P}_a$ and represent any element of $\mathcal{P}$ by writing it as a (strict) convex combination of such extreme points. Verify that this coincides with the answer found in (b).