Introduction to Mathematical Finance

Exercise sheet 3

Please submit your solutions online until Wednesday 22:00, 13/03/2024.

Exercise 3.1 For a market $(\mathcal{D}, \pi)$ with a numéraire D^0, a *martingale measure* for numéraire D^0 is a probability measure $\mathbb{Q}$ on $\mathcal{F}$ with $E_{\mathbb{Q}}[\frac{D^l}{D^0}] = \pi^l$ for $l = 0, 1, ..., N$. We call $\mathbb{Q}$ *equivalent* to $\mathbb{P}$ if $\mathbb{Q}[\{\omega_k\}] > 0$ for $k = 1, ..., K$, and absolutely continuous with respect to P if $Q[\{\omega_k\}] \geq 0$ for $k = 1, ..., K$. Denote by $\mathcal{P}$ (resp. $\mathcal{P}_a$) the set of all equivalent (resp. absolutely continuous) martingale measures for the numéraire D^0. Consider an arbitrage-free market with numéraire D^0.

(a) Show that $\mathcal{P}_a = \overline{\mathcal{P}}$. Here we identity $\mathcal{P}$ with a subset of $\mathbb{R}^N_+$ and denote by $\overline{}$ the closure in $\mathbb{R}^N$.

(b) Use (a) to show that for any random variable X,

$$\sup_{\mathbb{Q} \in \mathcal{P}} E_{\mathbb{Q}}[X] = \sup_{\mathbb{Q} \in \mathcal{P}_a} E_{\mathbb{Q}}[X].$$

(c) Show that for any payoff H, the supremum

$$\sup_{\mathbb{Q} \in \mathcal{P}_a} E_{\mathbb{Q}}\left[\frac{H}{D^0}\right]$$

is attained in some $\mathbb{Q} \in \mathcal{P}_a$. Does this imply that the market is complete?

Solution 3.1

(a) Let $R \in \mathbb{P}_a$, $Q \in \mathbb{P}$ and define $Q_\epsilon = \epsilon Q + (1 - \epsilon)R$ for $\epsilon \in (0, 1]$. Then since $Q_\epsilon \in \mathbb{P}$ and $Q_\epsilon \to R$ as $\epsilon \searrow 0$, R is the limit of a sequence in $\mathbb{P}$. Since R is arbitrary, we conclude that $\overline{\mathbb{P}} = \mathbb{P}_a$.

(b) Since $\bar{\mathbb{P}} = \mathbb{P}_a$, the inequality $\sup_{Q \in \mathbb{P}} E_Q[X] \leq \sup_{Q \in \mathbb{P}_a} E_Q[X]$ is trivial.

Since $\mathbb{P}$ is a bounded subset of $\mathbb{R}^K$, the set $\mathbb{P}_a = \bar{\mathbb{P}}$ is compact and the mapping $Q \mapsto E_Q[X]$ is linear, hence continuous, there exists an element $Q^* \in \text{argmax}_{Q \in \mathbb{P}_a} E_Q[X]$. Let Q be an arbitrary element in $\mathbb{P}$ and construct $Q^\epsilon = \epsilon Q + (1 - \epsilon) Q^*$. Then $Q^\epsilon \in \mathbb{P}$ for all $\epsilon \in (0, 1]$ by construction, and

$$\lim_{\epsilon \searrow 0} E_{Q^\epsilon}[X] = \lim_{\epsilon \searrow 0} (\epsilon E_Q[X] + (1 - \epsilon) E_{Q^*}[X]) = E_{Q^*}[X].$$

From the last equality, it follows that

$$\sup_{Q \in \mathbb{P}} E_Q[X] \geq \lim_{\epsilon \searrow 0} E_{Q^\epsilon}[X] \geq \sup_{Q \in \mathbb{P}_a} E_Q[X].$$

(c) Since the set $\mathbb{P}_a$ is compact and the mapping $Q \mapsto E_Q[\frac{H}{D^0}]$ is continuous, the supremum is in fact a maximum, hence attained.

However, this does not imply that H is attainable. If it did, then any arbitrage-free market would be complete, since H is arbitrary.

Exercise 3.2 Let

$$\pi = \begin{pmatrix} 1 \\ 1100 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1.1 & 1320 \\ 1.1 & 1210 \\ 1.1 & 880 \end{pmatrix}.$$

Denote by H the payoff of a put option with strike $K = 900$, i.e.

$$H = (900 - D^1)^+ = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix}.$$

(a) Find

$$\sup_{\mathbb{Q} \in \mathcal{P}} E_{\mathbb{Q}} \left[\frac{H}{D^0} \right].$$

(b) Compute

$$\inf\{\pi \cdot \vartheta : \vartheta \text{ with } \mathcal{D}\vartheta \geq H\}.$$

(c) Construct a market with $\mathcal{P}_a \neq \overline{\mathcal{P}}$.

Solution 3.2 Note that $\mathcal{D}$ is of the form

$$\mathcal{D} = \begin{pmatrix} \pi^0(1+r) & \pi^1(1+u) \\ \pi^0(1+r) & \pi^1(1+m) \\ \pi^0(1+r) & \pi^1(1+d) \end{pmatrix},$$

with $r = 0.1$, $u = 0.2$, $m = 0.1$ and $d = -0.2$.

(a) From Exercise 3.2, we know that

$$\mathbb{P} = \left\{ \left(\frac{(r-d) - (m-d)\lambda}{u-d}, \lambda, \frac{(u-r) - (u-m)\lambda}{u-d} \right) : \right.$$
$$\left. \lambda \in \left(0, \min\left\{ \frac{r-d}{m-d}, \frac{u-r}{u-m} \right\} \right) \right\}.$$

.

Since the expectation only depends on q_d, which is decreasing in λ, we find the supremum by setting $\lambda = 0$ to obtain

$$\frac{20}{1.1}\frac{1}{4} = \frac{5}{1.1}.$$

(b) Writing down the condition $\mathcal{D}\vartheta \geq H$, we obtain the optimization problem

$$\begin{aligned} \min \quad & \vartheta^0 + \pi^1\vartheta^1 \\ \text{s.t.} \quad & 11\vartheta^0 + 12\pi^1\vartheta^1 \geq 0, \\ & \vartheta^0 + \pi^1\vartheta^1 \geq 0, \\ & 11\vartheta^0 + 8\pi^1\vartheta^1 \geq 10H_3 = 200. \end{aligned}$$

Note that the second condition cannot be satisfied with equality without violating the other two, so that the solution must be found by solving the two extreme conditions with equality. Therefore,

$$\vartheta^s = \begin{pmatrix} \frac{30}{11} \\ -\frac{5}{2\pi^1} \end{pmatrix} H_3$$

is a solution to the optimization problem, and $\pi \cdot \vartheta^s = \frac{5}{1.1}$.

(c) Consider a market

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The only measure Q satisfying the martingale property is identified by $q = (0, 1)^{tr}$, and is not equivalent to P.

Exercise 3.3 Consider the one-step *trinomial model* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$ and u, m, d with $u > m > d$ and $u > r > d$.

(a) Show that $\mathcal{P} \neq \emptyset$.

(b) Describe the set $\mathcal{P}$.

Hint: Use the $\mathbb{Q}-$probability of the 'middle outcome' as a parameter in a parametrization of $\mathcal{P}$ as a line segment in $\mathbb{R}^3$.

(c) Denote by $\mathcal{P}_a$ the set of all martingale measures $\mathbb{Q}$ which are absolutely continuous with respect to $\mathbb{P}$. An element $\mathbb{R} \in \mathcal{P}_a$ is an *extreme point* if $\mathbb{R}$ cannot be written as a strict convex combination of elements in $\mathcal{P}_a$, i.e. the condition $\mathbb{R} = \lambda\mathbb{Q} + (1-\lambda)\mathbb{Q}'$ with $0 < \lambda < 1$ and both $\mathbb{Q}, \mathbb{Q}' \in \mathcal{P}_a$ implies that $\mathbb{Q} = \mathbb{Q}'$. Find the extreme points of $\mathcal{P}_a$ and represent any element of $\mathcal{P}$ by writing it as a (strict) convex combination of such extreme points. Verify that this coincides with the answer found in (b).

Solution 3.3

(a) Any $Q \in \mathbb{P}$ can be identified with a vector $q \in \mathbb{R}^3_{++}$ satisfying $\mathcal{D}^{tr} q = (1+r)\pi$. This is a system of 2 equations

$$(1+r)(q_1 + q_2 + q_3) = 1 + r,$$

$$(1+u)q_1 + (1+m)q_2 + (1+d)q_3 = 1 + r,$$

which reduces to

$$q_1 + q_2 + q_3 = 1, \quad uq_1 + mq_2 + dq_3 = r.$$

As u>m>d, this can be solved with $q \in \mathbb{R}^3_{++}$ if and only if r is a strict convex combination of u, m and d, which is equivalent to $u > r > d$ because m is already a strict convex combination of u and d.

(b) Let Q be any probability measure on $\mathcal{F}$ and $q_i = Q[\{\omega_i\}]$ for $i \in \{u, m, d\}$. Now write down the conditions on q_i; they are

$$\begin{aligned} 1 = \pi^1 = E_Q\left[\frac{D^1}{D^0}\right], \\ = \frac{(1+u)q_u + (1+m)q_m + (1+d)q_d}{1+r}, \end{aligned} \tag{1}$$

$$1 = q_u + q_m + q_d, \tag{2}$$

$$q_i \in (0,1), \quad i \in \{u, m, d\}. \tag{3}$$

Condition (1) follows from the martingale property, condition (2) follows from $Q[\Omega] = 1$, and condition (3) follows from $Q \approx P$. As suggested in the hint, we parametrize this set by choosing $q_m = \lambda$. Using the two equations then yields

$$q_u = \frac{(r-d)-(m-d)\lambda}{u-d},$$
$$q_d = \frac{(u-r)-(u-m)\lambda}{u-d}.$$

Now we just have to restrict λ according to the third condition. This amounts to choosing λ such that

$$q_m \in (0,1) \Leftrightarrow \lambda \in (0,1),$$
$$q_u \in (0,1) \Leftrightarrow \lambda \in \left(0, \frac{r-d}{m-d}\right),$$
$$q_d \in (0,1) \Leftrightarrow \lambda \in \left(0, \frac{u-r}{u-m}\right).$$

Since $u > m > d$ and $u > r > d$, this reduces to

$$\lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right).$$

Hence the identification of $\mathbb{P}$ as a subset of $[0,1]^3$ is given by

$$\mathbb{P} = \left\{ \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{(u-r)-(u-m)\lambda}{u-d}\right) : \lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right) \right\}.$$

(c) The extreme points can be found in two ways. First, one could calculate $\mathbb{P}_a$ explicitly to obtain the closure of $\mathbb{P}$ found above and setting the parameter λ to its smallest and largest values. More precisely,

$$\mathbb{P}_a = \left\{ \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{(u-r)-(u-m)\lambda}{u-d}\right) : \lambda \in \left[0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right] \right\}.$$

Alternatively, since $\mathbb{P}_a$ is the intersection of the two planes given by (2) and (1) in the closed, positive orthant of $\mathbb{R}^3$, the extreme points lie on the boundary of this orthant. Therefore, one could find solutions to (1) and (2) with $q_i \geq 0$ for all $i \in \{u, m, d\}$ and $q_i = 0$ for at least one $i \in \{u, m, d\}$. These points are given by

$$Q_1 = \left(\frac{r-d}{u-d}, 0, \frac{u-r}{u-d}\right) \quad \text{and} \quad Q_2 = \begin{cases} \left(0, \frac{r-d}{m-d}, \frac{m-r}{m-d}\right) & \text{if } m \geq r, \\ \left(\frac{r-m}{u-m}, \frac{u-r}{u-m}, 0\right) & \text{if } m < r. \end{cases}$$

Therefore,

$$\mathbb{P} = \{(1-\alpha)Q_1 + \alpha Q_2 : \alpha \in (0,1)\}.$$

We verify for $m \geq r$. The other case works analogously. For these parameters,

$$\lambda \in \left(0, \frac{r-d}{m-d}\right).$$

Let $\alpha = \lambda \frac{m-d}{r-d}$. Then $\alpha \in (0,1)$ and any element in $\mathbb{P}$ is given by

$$\begin{aligned}
(1-\alpha)Q_1 + \alpha Q_2 &= \left(\frac{(r-d)-(r-d)\alpha}{u-d}, \frac{r-d}{m-d}\alpha, \frac{(u-r)-(u-r)\alpha}{u-d} + \frac{m-r}{m-d}\alpha\right) \\
&= \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{u-r}{u-d} + \underbrace{\left(\frac{m-r}{m-d} - \frac{u-r}{u-d}\right)}_{-\frac{u-m}{u-d}\frac{r-d}{m-d}}\alpha\right) \\
&= \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{u-r-(u-m)\lambda}{u-d}\right),
\end{aligned}$$

which is of the same form as in the previous exercise.

Note: When $m \neq r$, Q_1 and Q_2 are both EMMs for the binomial markets obtained when one of the three points is removed.