Introduction to Mathematical Finance

Exercise sheet 4

Please submit your solutions online until Wednesday 22:00, 20/03/2024.

Exercise 4.1

Consider a model (with a numéraire) with $d = 1$ traded risky asset X, with $X_0 = 1$ and

$$\Delta X_t = \eta_t, \qquad t = 1, 2, 3,$$

where the η_t are i.i.d. $\eta_1 \sim \mathcal{N}(0,1)$-distributed.

(a) Suppose that a trader decides at time $t = 0$ to buy 2 shares, to sell 3 shares at $t = 1$ and then to buy 1 share at time $t = 2$. Denote by G_t his cumulative gain from the corresponding self-financing trading strategy. Find the distribution of G_3.

(b) Suppose that $\mathcal{F}_t = \sigma(S_1, \dots, S_t)$ for $t = 1, 2, 3$. Show that there is no arbitrage in this model.

Solution 4.1

(a) Recall the following property of the Gaussian distribution: if $Y \sim \mathcal{N}(a, b)$, $Z \sim \mathcal{N}(c, d)$ and Y, Z are independent, then for any $\alpha, \beta \in \mathbb{R}$, we have $\alpha Y + \beta Z \sim \mathcal{N}(\alpha a + \beta c, \alpha^2 b + \beta^2 d)$. Thus $G_3 = 2\Delta X_1 - 3\Delta X_2 + \Delta X_3 \sim \mathcal{N}(0, 14)$.

(b) From the definition of the model, we have $E[(X_t - X_{t-1}) | \mathcal{F}_{t-1}] = 0$, $=$ so that X is a martingale. Hence by Proposition II.2.3, there is no arbitrage.

Exercise 4.2

Consider a market with trading dates $t = 0, \dots, T$, with N traded assets on the probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,\dots,T}$, i.e., a general multiperiod market.

For any strategy ψ, we define the process $\widetilde{C}(\psi) = (\widetilde{C}_t)_{t=0,\dots,T}(\psi)$ by

$$\widetilde{C}_t(\psi) := \widetilde{V}_t(\psi) - \widetilde{G}_t(\psi).$$

The process $\widetilde{C}$ is called the *cost process* for ψ.

(a) Show that

$$\Delta\widetilde{C}_{t+1}(\psi) = \Delta\psi_{t+1} \cdot S_t,$$

for $t = 1, \dots, T-1$.

(b) Show that ψ is self-financing if and only if

$$\widetilde{C}_t(\psi) = \widetilde{C}_0(\psi)$$

for $t = 1, \dots, T$.

Hint: Be careful with definitions at the first time point.

Solution 4.2

(a) We need to show that $\Delta\widetilde{V}_{t+1}(\psi) - \Delta\widetilde{G}_{t+1}(\psi) = \Delta\psi_{t+1} \cdot S_t$ for $t = 1, \dots, T-1$. By the definitions,

$$\begin{aligned} \Delta\widetilde{V}_{t+1}(\psi) - \Delta\widetilde{G}_{t+1}(\psi) &= \psi_{t+1} \cdot S_{t+1} - \psi_t \cdot S_t - \psi_{t+1} \cdot \Delta S_{t+1} \\ &= -\psi_t \cdot S_t + \psi_{t+1} \cdot S_t \\ &= \Delta\psi_{t+1} \cdot S_t. \end{aligned}$$

(b) The property $\widetilde{C}_t(\psi) = \widetilde{C}_0(\psi)$ for $t = 1, \dots, T$ is equivalent to

$$\Delta\widetilde{C}_{t+1}(\psi) = 0$$

for $t = 0, \dots, T-1$.

As we can see from (a), this condition is stronger than ψ being self-financing. However, we claim that $\Delta\widetilde{C}_1(\psi) = 0$ always holds. Indeed,

$$\widetilde{C}_1(\psi) = \widetilde{V}_1(\psi) - \widetilde{G}_1(\psi) = \psi_1 \cdot S_1 - \psi_1 \cdot \Delta S_1 = \psi_1 \cdot S_0 = \widetilde{V}_0(\psi) = \widetilde{C}_0(\psi).$$

Combining this observation with (a), the definition of ψ being self-financing is equivalent to $\Delta\widetilde{C}_{t+1}(\psi) = 0$ for $t = 0, \dots, T-1$, which in turn is equivalent to $\widetilde{C}_t(\psi) = \widetilde{C}_0(\psi)$ for $t = 1, \dots, T$.

Exercise 4.3 Consider the standard model for a financial market in finite discrete time with a numéraire S^0.

(a) Show that a strategy ψ is self-financing for S if and only if it is self-financing for S/S^0.

(b) Show that S satisfies NA if and only if S/S^0 satisfies NA.

Solution 4.3

(a) Notice that for $k = 1, \dots, T-1$

$$(\psi_{k+1} - \psi_k)S_k = 0 \text{ if and only if } (\psi_{k+1} - \psi_k)S_k/S_k^0 = 0.$$

That means that a strategy ψ is self-financing for S if and only if it is self-financing for S/S^0.

(b) We know that for a numéraire $S^0 > 0$ we have $V(\psi) = \widetilde{V}(\psi)/S^0$, so that $\{-\widetilde{V}_0(\psi) \geq 0 \text{ a.s.}, \widetilde{V}_T(\psi) \geq 0 \text{ a.s.}\}$ if and only if $\{-V_0(\psi) \geq 0 \text{ a.s.}, V_T(\psi) \geq 0 \text{ a.s.}\}$ Thus we conclude that existence of (generalized) arbitrage for S is equivalent to the existence of arbitrage for S/S^0.

Claim. NA for $S \Leftrightarrow$ NA' for S.

Clear that NA' for $S/S^0 \Rightarrow$ NA for S/S^0. To show the converse, we will use the following strategy: NA for $S/S^0 \Rightarrow$ NA for $S \Rightarrow$ NA' for $S \Rightarrow$ NA' for S/S^0. The only implication here to prove is "NA for $S/S^0 \Rightarrow$ NA for S".

Suppose that NA for S/S^0 holds. Consider the construction in "4) $\Rightarrow$ 5)" of P II.2.1. This constructs $\bar{\psi}$, self-financing, with $V_0(\bar{\psi}) = 0$ P-a.s, $V_T(\bar{\psi}) \in L^0_+ \setminus \{0\}$, and $V(\bar{\psi}) \geq 0$ i.e. an arbitrage opportunity of first kind for S/S^0 which is 0-admissible for S/S^0.

Multiplying everywhere by $S^0 > 0$ gives $\widetilde{V}_0(\bar{\psi}) = 0$ P-a.s., $\widetilde{V}_T(\bar{\psi}) \in L^0_+ \setminus \{0\}$ and $\widetilde{V}(\bar{\psi}) \geq 0$, i.e., an arbitrage of first kind for S which is 0-admissible for S. This is exactly what we want.