Coordinator
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Introduction to Mathematical Finance

Exercise sheet 5

Please submit your solutions online until Wednesday 10pm, 27/03/2024.

Exercise 5.1 Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F} = \sigma(A_1, \dots, A_n)$, where $\bigcup_{i=1}^n A_i = \Omega$ and $A_i \bigcap A_j = \emptyset$ for $i \neq j$. A probability measure $\mathbb{Q}$ on $\mathcal{F}$ is called absolutely continuous with respect to $\mathbb{P}$ if for any $A \in \mathcal{F}$, $\mathbb{P}[A] = 0$ implies that $\mathbb{Q}[A] = 0$.

(a) Show directly, without using the Radon–Nikodym theorem, that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ if and only if there exists a random variable $\xi \geq 0$ with $E^{\mathbb{P}}[\xi] = 1$ and

$$\mathbb{Q}[A] = \int_A \xi d\mathbb{P} \quad \text{for all } A \in \mathcal{F}.$$

(b) Two probability measures $\mathbb{Q}$ and $\mathbb{P}$ on $\mathcal{F}$ are equivalent on $\mathcal{F}$ if for any $A \in \mathcal{F}$, we have $\mathbb{Q}[A] = 0$ if and only if $\mathbb{P}[A] = 0$. Construct an example where $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, but $\mathbb{Q}$ and $\mathbb{P}$ are not equivalent.

Exercise 5.2 Let $\psi = (V_0, \vartheta)$ be a self-financing strategy in a multi-period market with discounted asset prices. Assume that $V_T(\psi) \geq -a$ P-a.s. for some $a \geq 0$.

(a) Show that if the market is arbitrage-free, then ψ is a-admissible for S/S^0, i.e., $V_t(\psi) \geq -a$ P-a.s. for all $t = 0, \dots, T$.

(b) Show without using (a) that if X admits an ELMM Q and $V_0 \in L^1(Q)$, then $V_t(\psi) \geq -a$ P-a.s. for all $t = 0, \dots, T$.

Exercise 5.3 Let M be a local martingale which is bounded from below by $-a$ for some $a \geq 0$ and is integrable at the initial time, i.e., $M_0 \in L^1(P)$. Show from the definitions that M is a supermartingale.