Introduction to Mathematical Finance

Exercise sheet 5

Please submit your solutions online until Wednesday 10pm, 27/03/2024.

Exercise 5.1 Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F} = \sigma(A_1, \ldots, A_n)$, where $\bigcup_{i=1}^n A_i = \Omega$ and $A_i \bigcap A_j = \emptyset$ for $i \neq j$. A probability measure $\mathbb{Q}$ on $\mathcal{F}$ is called absolutely continuous with respect to $\mathbb{P}$ if for any $A \in \mathcal{F}$, $\mathbb{P}[A] = 0$ implies that $\mathbb{Q}[A] = 0$.

(a) Show directly, without using the Radon–Nikodym theorem, that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ if and only if there exists a random variable $\xi \geq 0$ with $E^{\mathbb{P}}[\xi] = 1$ and

$$\mathbb{Q}[A] = \int_A \xi d\mathbb{P} \quad \text{for all } A \in \mathcal{F}.$$

(b) Two probability measures $\mathbb{Q}$ and $\mathbb{P}$ on $\mathcal{F}$ are equivalent on $\mathcal{F}$ if for any $A \in \mathcal{F}$, we have $\mathbb{Q}[A] = 0$ if and only if $\mathbb{P}[A] = 0$. Construct an example where $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, but $\mathbb{Q}$ and $\mathbb{P}$ are not equivalent.

Solution 5.1

(a) Consider first Q defined by $Q[A] = \int_A \xi dP$ for $A \in \mathcal{F}$. From the definition of $\mathcal{F}$ it follows that any set $A \in \mathcal{F}$ is of the form $A = \bigcup_{j \in J} A_j$, where $J \subseteq \{1, ..., n\}$. So that for any $A \in \mathcal{F}$ $Q[A] = \sum_{j \in J} Q[A_j]$. From $\xi \geq 0$ and $E^P[\xi] = 1$ it is clear that Q is a probability measure on $\mathcal{F}$. Since ξ is a random variable on $(\Omega, \mathcal{F})$, it is of the form $\xi = \sum_{i=1}^n c_i I_{A_i}$ for some $c_i \geq 0$. Since for any $A \in \mathcal{F}$ $Q[A] = \sum_{j \in J} Q[A_j] = \sum_{j \in J} c_j P[A_j]$ and $P[A] = \sum_{j \in J} P[A_j]$, then $P[A] = 0$ implies that $Q[A] = 0$, so that Q is absolutely continuous with respect to P on $\mathcal{F}$.

Now suppose that Q is absolutely continuous with respect to P on $\mathcal{F}$. For $i \in \{1, \ldots, n\}$ if $P[A_i] = 0$ define $c_i := 0$ and otherwise $c_i := \frac{Q[A_i]}{P[A_i]}$. Then $\xi := \sum c_i I_{A_i}$ is clearly ≥ 0, and the construction of ξ implies that $E^P[\xi] = \sum c_i P[A_i] = \sum Q[A_i] = 1$ and $Q[A_i] = \int_{A_i} \xi dP$.

(b) Consider $\mathcal{F} = \sigma(A_1, A_2)$, with $A_1 \cup A_2 = \Omega$ and $A_1 \bigcap A_2 = \emptyset$. Consider the probability measures Q and P defined by $Q[A_1] = 1, Q[A_2] = 0$ and $P[A_1] = P[A_2] = 1/2$. Then it is clear that Q is absolutely continuous with respect to P, but P and Q are not equivalent.

Exercise 5.2 Let $\psi = (V_0, \vartheta)$ be a self-financing strategy in a multi-period market with discounted asset prices. Assume that $V_T(\psi) \geq -a$ P-a.s. for some $a \geq 0$.

(a) Show that if the market is arbitrage-free, then ψ is a-admissible for S/S^0, i.e., $V_t(\psi) \geq -a$ P-a.s. for all $t = 0, \dots, T$.

(b) Show without using (a) that if X admits an ELMM Q and $V_0 \in L^1(Q)$, then $V_t(\psi) \geq -a$ P-a.s. for all $t = 0, \dots, T$.

Solution 5.2

(a) Suppose there exists a time point k such that the event $A = \{V_k(\psi) < -a\}$ has a strictly positive probability, i.e., $P[A] > 0$. Let k_0 be the largest such time point. Construct a self-financing strategy $\bar{\psi} = (\bar{\psi}^0, \bar{\vartheta})$ described by $V_0 = 0$ and

$$\bar{\vartheta}_k = \begin{cases} 0 & \text{if } k \neq k_0 + 1, \\ \vartheta_k I_A & \text{if } k = k_0 + 1. \end{cases}$$

Note that this process is indeed predictable since ϑ is predictable and $A \in \mathcal{F}_{k_0}$, and that $\bar{\psi}$ defines a self-financing strategy by Proposition II.1.2. Then $V(\bar{\psi}) = V_0 + G(\bar{\vartheta})$ implies that

$$\begin{aligned} V_T(\bar{\psi}) &= V_{k_0+1}(\bar{\psi}) = V_{k_0} + \Delta G_{k_0+1}(\bar{\vartheta}) \\ &= 0 + I_A \Delta G_{k_0+1}(\vartheta) = \underbrace{V_{k_0+1}(\psi) I_A}_{\geq -a I_A} - \underbrace{V_{k_0}(\psi) I_A}_{< -a I_A} \geq 0 \quad P\text{-a.s.}, \end{aligned}$$

with strict inequality on A. In other words, $\bar{\psi}$ is an arbitrage opportunity. This is a contradiction, whence we conclude that ψ is a-admissible.

(b) By assumption, X is a local Q-martingale. Therefore, by Proposition C.4, $G(\vartheta)$ and hence also $V(\psi)$ is a local Q-martingale. Furthermore,

$$E_Q[|V_0(\psi)|] < \infty$$

and $E_Q[V_T^-(\psi)] \leq a$. So from Theorem C.5 we conclude that $V(\psi)$ is a true Q-martingale.

By the martingale property,

$$V_k(\psi) = E[V_T(\psi) | \mathcal{F}_k] \geq -a \quad Q\text{-a.s.},$$

for all $t = 0, \dots, T$, thus also P-a.s., which is what we wanted to show.

Exercise 5.3 Let M be a local martingale which is bounded from below by $-a$ for some $a \geq 0$ and is integrable at the initial time, i.e., $M_0 \in L^1(P)$. Show from the definitions that M is a supermartingale.

Solution 5.3 Denote by $(\tau_n)_{n \in \mathbb{N}}$ a localising sequence and let $Y = M + a$ so that $Y \geq 0$ P-a.s. Then $Y^{\tau_n} I_{\{\tau_n > 0\}}$ is a nonnegative martingale for every $n \in \mathbb{N}$. Note that since Y_0 is integrable, we can even drop the indicator function. Indeed,

$$Y_k^{\tau_n} = Y_0 I_{\{\tau_n = 0\}} + Y_k^{\tau_n} I_{\{\tau_n > 0\}},$$

where the first term is integrable, $\mathcal{F}_0$-measurable and constant in k, hence a martingale. Moreover, since $\tau_n \nearrow \infty$ P-a.s.,

$$\lim_{n \to \infty} Y_k^{\tau_n} = Y_k \quad P\text{-a.s.}$$

for every $k \in \mathbb{N}_0$. Now M is adapted by assumption, and by Fatou's lemma,

$$E\left[\liminf_{n \to \infty} Y_k^{\tau_n}\right] \leq \liminf_{n \to \infty} E[Y_k^{\tau_n}] = \liminf_{n \to \infty} E[Y_0^{\tau_n}] = E[Y_0] < \infty.$$

As $Y \geq 0$, this shows that Y is integrable, and so is then $M = Y - a$. Finally, using the conditional version of Fatou's lemma gives

$$\begin{aligned} E[Y_k|\mathcal{F}_j] &= E\left[\liminf_{n \to \infty} Y_k^{\tau_n} \middle| \mathcal{F}_j\right] \\ &\leq \liminf_{n \to \infty} E[Y_k^{\tau_n}|\mathcal{F}_j] = \liminf_{n \to \infty} Y_j^{\tau_n} = Y_j \quad P\text{-a.s.} \end{aligned}$$

for all $j \leq k$. This shows that Y, and therefore also M, is indeed a supermartingale.

Remark. The preceding arguments do not use anywhere that M is indexed by discrete time. So the result also holds in continuous time.