Coordinator
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Introduction to Mathematical Finance

Exercise sheet 6

Please submit your solutions online until Wednesday 10pm, 10/04/2024.

Exercise 6.1

(a) Suppose that $(\xi_k)_{k\in\mathbb{N}}$ are independent integrable random variables with expectation 1. Define the process $X = \{X_n\}_{n\in\mathbb{N}_0}$ by $X_n := \prod_{k=1}^{n} \xi_k$. Show that X is a martingale for its natural filtration.

(b) Give an example of a stochastic process in discrete time which is not locally bounded.

Exercise 6.2

Consider a sequence $(\xi_k)_{k\in\mathbb{N}}$ of i.i.d. random variables with $\xi_1 \sim \mathcal{N}(0,1)$. Define the process $M = (M_n)_{n\in\mathbb{N}_0}$ by $M_n := \sum_{k=1}^{n} \xi_k$. Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be the natural filtration of M.

(a) Show that $X_n := M_n^2 - n, n \in \mathbb{N}_0$, is a martingale.

(b) Show that $Y_n := \exp(M_n - n/2), n \in \mathbb{N}_0$, is a martingale.

(c) For any bounded predictable process $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ define $N := \alpha \cdot M$ so that $N_k = \sum_{i=1}^{k} \alpha_i(M_i - M_{i-1})$ for $k \in \mathbb{N}_0$. Define also $\langle N \rangle = (\langle N \rangle_k)_{k\in\mathbb{N}_0}$ by $\langle N \rangle_k := \sum_{i=1}^{k} \alpha_i^2$. Show that $X := N^2 - \langle N \rangle$ and $Y := \exp(N - \langle N \rangle/2)$ are martingales.

Exercise 6.3

Using the notions from the lecture, show that the following are equivalent:

(a) $S = S^0(1, X)$ satisfies NA.

(b) $\mathcal{G}_{adm} \bigcap L^0_+ = \{0\}$.

(c) $\mathcal{C}_{adm} \bigcap L^0_+ = \{0\}$.