Introduction to Mathematical Finance

Exercise sheet 6

Please submit your solutions online until Wednesday 10pm, 10/04/2024.

Exercise 6.1

(a) Suppose that $(\xi_k)_{k\in\mathbb{N}}$ are independent integrable random variables with expectation 1. Define the process $X = \{X_n\}_{n\in\mathbb{N}_0}$ by $X_n := \prod_{k=1}^{n} \xi_k$. Show that X is a martingale for its natural filtration.

(b) Give an example of a stochastic process in discrete time which is not locally bounded.

Solution 6.1

(a) Denote the natural filtration of X as $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$. Then X is adapted by definition, and integrable because the product of independent integrable random variables is integrable. Moreover, by the definition of X and the properties of conditional expectation, we have

$$E[X_n - X_{n-1}|\mathcal{F}_{n-1}] = \Big(\prod_{k=1}^{n-1} \xi_k\Big) E[\xi_n - 1|\mathcal{F}_{n-1}] = \Big(\prod_{k=1}^{n-1} \xi_k\Big) E[\xi_n - 1] = 0.$$

Thus X is a martingale.

(b) Start with a sequence $(\xi_k)_{k\in\mathbb{N}}$ of nonnegative random variables and define the process $X = (X_n)_{n\in\mathbb{N}_0}$ by $X_n = \sum_{k=1}^{n} \xi_k$. Then $X_0 = 0$, so $X^\tau = X^\tau I_{\{\tau>0\}}$, and so it is enough to consider any stopping times $\tau \geq 1$. Then $X_\tau \geq X_1 I_{\{\tau\geq 1\}} = \xi_1$ because the ξ_k are nonnegative. So if ξ_1 is unbounded, then X^τ cannot be bounded for any stopping time $\tau \geq 1$, and so X is not locally bounded.

Exercise 6.2

Consider a sequence $(\xi_k)_{k\in\mathbb{N}}$ of i.i.d. random variables with $\xi_1 \sim \mathcal{N}(0,1)$. Define the process $M = (M_n)_{n\in\mathbb{N}_0}$ by $M_n := \sum_{k=1}^n \xi_k$. Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be the natural filtration of M.

(a) Show that $X_n := M_n^2 - n, n \in \mathbb{N}_0$, is a martingale.

(b) Show that $Y_n := \exp(M_n - n/2), n \in \mathbb{N}_0$, is a martingale.

(c) For any bounded predictable process $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ define $N := \alpha \cdot M$ so that $N_k = \sum\limits_{i=1}^{k} \alpha_i(M_i - M_{i-1})$ for $k \in \mathbb{N}_0$. Define also $\langle N\rangle = (\langle N\rangle_k)_{k\in\mathbb{N}_0}$ by $\langle N\rangle_k := \sum\limits_{i=1}^{k} \alpha_i^2$. Show that $X := N^2 - \langle N\rangle$ and $Y := \exp(N - \langle N\rangle/2)$ are martingales.

Solution 6.2

Questions (a) and (b) follow directly from (c) by taking $\alpha \equiv 1$. So we only prove (c). Clearly, N is adapted and $\langle N\rangle$ is predictable for $\mathbb{F}$. Moreover, both X and Y are integrable because α is bounded and the ξ_i have all exponential moments. For the martingale property, note that N is a martingale, so that

$$E[\Delta(N^2)_k|\mathcal{F}_{k-1}] = E[(\Delta N_k)^2|\mathcal{F}_{k-1}] =$$

$$= \alpha_k^2 E[(\Delta M_k)^2|\mathcal{F}_{k-1}] = \alpha_k^2 E[\xi_k^2] = \alpha_k^2 = \Delta\langle N\rangle_k.$$

This shows that $X = N^2 - \langle N\rangle$ is a martingale.

Similarly, using that α_k is $\mathcal{F}_{k-1}$ - measurable, ξ_k is independent of $\mathcal{F}_{k-1}$ and $\Delta N_k = \alpha_k \Delta M_k = \alpha_k \xi_k$, we get $E[Y_k/Y_{k-1}|\mathcal{F}_{k-1}] = E[\exp(\Delta N_k - \Delta\langle N\rangle_k/2)|\mathcal{F}_{k-1}] = E[\exp(\alpha_k\xi_k - \alpha_k^2/2)|\mathcal{F}_{k-1}] = E[\exp(\lambda\xi_k - \lambda^2/2)]|_{\lambda=\alpha_k} = 1$ because $\xi_k \sim \mathcal{N}(0,1)$. So Y is also a martingale.

Exercise 6.3

Using the notions from the lecture, show that the following are equivalent:

(a) $S = S^0(1, X)$ satisfies NA.

(b) $\mathcal{G}_{adm} \bigcap L^0_+ = \{0\}$.

(c) $\mathcal{C}_{adm} \bigcap L^0_+ = \{0\}$.

Solution 6.3 By Exercise 4.3(c), NA for S and NA for S/S^0 are equivalent. By Proposition II.2.1, NA for S/S^0 is equivalent to $\mathcal{G} \bigcap L^0_+(\mathcal{F}_T) = \{0\}$, and this is also equivalent to $\mathcal{G} \bigcap L^0_+ = \{0\}$ because $\mathcal{G} \subseteq L^0(\mathcal{F}_T)$. Because $0 \in \mathcal{G}_{adm} \subseteq \mathcal{G}$, the condition $\mathcal{G} \bigcap L^0_+ = \{0\}$ implies $\mathcal{G}_{adm} \bigcap L^0_+ = \{0\}$, and so we get "$(a) \Rightarrow (b)$".
Conversely, if we look at the proof of "5) $\Rightarrow$ 1)" for Proposition II.2.1, we can see that a slight modification also proves that $\mathcal{G}_{adm} \bigcap L^0_+ = \{0\}$ implies NA for S/S^0. Indeed, if ψ is a-admissible in that argument, then

$$V(\bar{\psi}) = V(\psi) - V_0(\psi)V(\psi^*) = V(\psi) - V_0(\psi) \geq V(\psi)$$

shows that $\bar{\psi}$ is also a-admissible, and the rest of the argument goes as before. So we also have "$(b) \Rightarrow (a)$".
Because $0 \in \mathcal{G}_{adm} \subseteq \mathcal{C}_{adm}$, we clearly have "$(c) \Rightarrow (b)$". Conversely, if $c \in \mathcal{C}_{adm}$, then $c = g - Y$ with $g \in \mathcal{G}_{adm}$ and $Y \geq 0$. If also $c \in L^0_+$, then $c \geq 0$ and $g = c + Y \geq 0$ so that $g \in \mathcal{G}_{adm} \bigcap L^0_+$. By (b), we then have $0 = g = c + Y$ with $c \geq 0, Y \geq 0$, and therefore also $c = 0$. This shows that $\mathcal{C}_{adm} \bigcap L^0_+ = \{0\}$ and hence "$(b) \Rightarrow (c)$".