Introduction to Mathematical Finance

Exercise sheet 7

Please submit your solutions online until Wednesday 10pm, 17/04/2024.

Exercise 7.1 Construct probability measures $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{Q} \overset{\text{loc}}{\approx} \mathbb{P}$ with $\mathbb{Q} \not\approx P$ and even $\lim\limits_{T\to\infty} Z_T^{\mathbb{Q};\mathbb{P}} = 0$ $\mathbb{P}$-a.s.

Exercise 7.2

(a) Let U be a standard normal random variable $U \sim \mathcal{N}(0,1)$. Consider a market with $T = 1, X_0 = 1$ and $X_1 = e^{\sigma U + \mu}$ for some constants $\mu, \sigma \in \mathbb{R}$, $\sigma \neq 0$. Construct an EMM for X.

(b) Consider a market with $X_0 = 1$ and $X_k := \prod\limits_{j=1}^{k} e^{R_j}$, $k = 1, \dots, T$, where $R_1, ..., R_T$ are i.i.d. with $R_1 \sim \mathcal{N}(\mu, \sigma^2)$ for some constants $\mu, \sigma \in \mathbb{R}$, $\sigma \neq 0$. Let $\mathbb{F} = (\mathcal{F}_t)_{t=1}^T$ be the natural filtration of X. Show that the market is arbitrage-free.

Exercise 7.3

Consider a market $(1, X)$ with $X_0 = 1$ and $X_k = \prod\limits_{j=1}^{k} R_j$ for $k = 1, ..., T$, where $R_1, ..., R_T$ are i.i.d. under $\mathbb{P}$ and > 0. The filtration $\mathbb{F}$ is generated by X. Suppose that we have an EMM $\mathbb{Q}$ for X of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{k=1}^{T} g_1(R_k)$$

for a measurable function $g_1 : (0, \infty) \mapsto (0, \infty)$. Show that $R_1, ..., R_T$ are also i.i.d. under $\mathbb{Q}$.