Introduction to Mathematical Finance

Exercise sheet 7

Please submit your solutions online until Wednesday 10pm, 17/04/2024.

Exercise 7.1 Construct probability measures $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{Q} \overset{\text{loc}}{\approx} \mathbb{P}$ with $\mathbb{Q} \not\approx P$ and even $\lim_{T\to\infty} Z_T^{\mathbb{Q};\mathbb{P}} = 0$ $\mathbb{P}$-a.s.

Solution 7.1

Let $\Omega = \{0,1\}^{\mathbb{N}}$, $Y_k(\omega) = \omega_k$, $\mathcal{F}_k = \sigma(Y_1,\dots,Y_k)$ and $\mathcal{F} = \mathcal{F}_\infty = \sigma(Y_1, Y_2, \dots)$. Let P and Q be measures with $P[Y_k = 1] = p \in (0,1)$, $Q[Y_k = 1] = q \in (0,1)$, $p \neq q$ and such that Y_k, $k \in \mathbb{N}$, are independent under both measures. Then

$$P[Y_1 = y_1, ..., Y_T = y_T] = p^{\sum_{i=1}^T y_i}(1-p)^{T-\sum_{i=1}^T y_i}$$

and

$$Q[Y_1 = y_1, ..., Y_T = y_T] = q^{\sum_{i=1}^T y_i}(1-q)^{T-\sum_{i=1}^T y_i}.$$

Define $S_T := \sum_{j=1}^T Y_j$. Then for $T \geq 1$

$$Z_T^{Q;P} = \frac{dQ}{dP}\bigg|_{\mathcal{F}_T} = \left(\frac{q}{p}\right)^{S_T}\left(\frac{1-q}{1-p}\right)^{T-S_T}.$$

Hence for every $k \in \mathbb{N}_0$, $P|_{\mathcal{F}_k} \approx Q|_{\mathcal{F}_k}$, and $P|_{\mathcal{F}_0} = Q|_{\mathcal{F}_0}$ because $\mathcal{F}_0 = \{\varnothing, \Omega\}$. However, by the law of large numbers, the set

$$A = \left\{\omega : \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \omega_k = p\right\}$$

has $P[A] = 1$, but $Q[A] = 0$. Hence, $P \not\approx Q$.

Suppose that $p > 1/2$ and $q = 1-p$. Then from the computations above we have $Z_T^{Q;P} = \left(\frac{q}{p}\right)^{2S_T - T} \longrightarrow 0$ as $T \to \infty$ P-a.s. by the law of large numbers.

Exercise 7.2

(a) Let U be a standard normal random variable $U \sim \mathcal{N}(0,1)$. Consider a market with $T = 1, X_0 = 1$ and $X_1 = e^{\sigma U + \mu}$ for some constants $\mu, \sigma \in \mathbb{R}$, $\sigma \neq 0$. Construct an EMM for X.

(b) Consider a market with $X_0 = 1$ and $X_k := \prod_{j=1}^{k} e^{R_j}$, $k = 1, \dots, T$, where $R_1, ..., R_T$ are i.i.d. with $R_1 \sim \mathcal{N}(\mu, \sigma^2)$ for some constants $\mu, \sigma \in \mathbb{R}$, $\sigma \neq 0$. Let $\mathbb{F} = (\mathcal{F}_t)_{t=1}^T$ be the natural filtration of X. Show that the market is arbitrage-free.

Solution 7.2

(a) It is enough to construct a positive random variable D with $E[D] = 1$ and $E[X_1 D] = 1$, because then we can define the EMM by $\frac{dQ}{dP} := D$. Consider D of the form $D = \exp(\alpha U + \beta)$ with some constants $\alpha, \beta \in \mathbb{R}$. Then we have $E[D] = \exp(\beta + \alpha^2/2)$; so $E[D] = 1$ if and only if $\beta = -\alpha^2/2$. From $E[X_1 D] = \exp(\mu + \beta + (\sigma + \alpha)^2/2)$, we conclude that $E[X_1 D] = 1$ if and only if $\mu + \beta + (\sigma + \alpha)^2/2 = 0$. Thus we have both $E[D] = 1$ and $E[X_1 D] = 1$ if and only if

$$\begin{cases} \beta = -\frac{\alpha^2}{2} \\ \mu + \beta + (\sigma + \alpha)^2/2 = 0 \end{cases},$$

which gives us, after substituting the first equation into the second, that

$$\begin{cases} \beta = -\frac{\alpha^2}{2} \\ \alpha = -\frac{\mu + \sigma^2/2}{\sigma} \end{cases}, \tag{1}$$

and $\frac{dQ}{dP} := D$ with these parameters defines the EMM for X.

(b) For $k = 1, \dots, T$, we have $X_k = \prod_{j=1}^{k} e^{R_j} = \prod_{j=1}^{k} e^{\sigma U_j + \mu}$, where $U_j := (R_j - \mu)/\sigma$ are i.i.d. with $U_1 \sim \mathcal{N}(0,1)$. Set $D_k := \exp(\alpha U_k + \beta)$, where α, β are defined in (a) by (1). Then with the same arguments as in (1) we have that $E[D_k | \mathcal{F}_{k-1}] = 1$ and $E[D_k \frac{X_k}{X_{k-1}} | \mathcal{F}_{k-1}] = E[D_k e^{R_k} | \mathcal{F}_{k-1}] = 1$. Hence

$$\frac{dQ}{dP} := \prod_{k=1}^{T} D_k$$

yields an EMM for X, and so the market is arbitrage-free.

Exercise 7.3

Consider a market $(1, X)$ with $X_0 = 1$ and $X_k = \prod_{j=1}^{k} R_j$ for $k = 1, ..., T$, where $R_1, ..., R_T$ are i.i.d. under $\mathbb{P}$ and > 0. The filtration $\mathbb{F}$ is generated by X. Suppose that we have an EMM $\mathbb{Q}$ for X of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{k=1}^{T} g_1(R_k)$$

for a measurable function $g_1 : (0, \infty) \mapsto (0, \infty)$. Show that $R_1, ..., R_T$ are also i.i.d. under $\mathbb{Q}$.

Solution 7.3 Random variables $Z_1, ..., Z_N$ are independent under P if and only if for any measurable and bounded functions $f_1, ..., f_N$, we have

$$E^P\Big[\prod_{j=1}^{N} f_j(Z_j)\Big] = \prod_{j=1}^{N} E^P\Big[f_j(Z_j)\Big].$$

Consider any measurable and bounded functions $f_1, ..., f_T$. Then

$$E^Q\left[\prod_{j=1}^{T} f_j(R_j)\right] = E^P\left[\prod_{j=1}^{T} f_j(R_j)\prod_{j=1}^{T} g_1(R_j)\right] = E^P\left[\prod_{j=1}^{T} f_j(R_j)g_1(R_j)\right].$$

Since $R_1, ..., R_T$ are independent under P, then for any measurable functions $z_1, ..., z_T$ such that $E^P[z_j(R_j)] < \infty$ for $j = 1, ..., T$, we have

$$E^P\left[\prod_{j=1}^{T} z_j(R_j)\right] = \prod_{j=1}^{T} E^P\left[z_j(R_j)\right].$$

Hence, by taking $z_j := f_j \cdot g_1$, we derive that

$$E^P\left[\prod_{j=1}^{T} f_j(R_j)g_1(R_j)\right] = \prod_{j=1}^{T} E^P\left[f_j(R_j)g_1(R_j)\right],$$

and so

$$E^Q\left[\prod_{j=1}^{T} f_j(R_j)\right] = \prod_{j=1}^{T} E^P\left[f_j(R_j)g_1(R_j)\right].$$

On the other hand , since $R_1, ..., R_T$ are i.i.d. under P, we have

$$1 = E^P\left[\frac{dQ}{dP}\right] = E^P\left[\prod_{j=1}^{T} g_1(R_j)\right] = \prod_{j=1}^{T} E^P\left[g_1(R_j)\right] = \Big(E^P\left[g_1(R_1)\right]\Big)^T,$$

hence $E^P\left[g_1(R_j)\right] = E^P\left[g_1(R_1)\right] = 1$. Finally,

$$\prod_{j=1}^{T} E^Q\left[f_j(R_j)\right] = \prod_{j=1}^{T} E^P\left[f_j(R_j)\prod_{j=1}^{T} g_1(R_j)\right] =$$

$$= \prod_{j=1}^{T} E^P \left[f_j(R_j) g_1(R_j) \right] \prod_{i \neq j} E^P \left[g_1(R_i) \right] = \prod_{j=1}^{T} E^P \left[f_j(R_j) g_1(R_j) \right].$$

Thus we proved that

$$E^Q \left[\prod_{j=1}^{T} f_j(R_j) \right] = \prod_{j=1}^{T} E^P \left[f_j(R_j) g_1(R_j) \right] = \prod_{j=1}^{T} E^Q \left[f_j(R_j) \right],$$

hence $R_1, ..., R_T$ are independent under Q.

Since $R_1, ..., R_T$ are i.i.d. under P, then for $j = 1, ..., T$ and any measurable and bounded function f, we have

$$E^Q[f(R_j)] = E^P[f(R_j) g_1(R_j)] \prod_{i \neq j} E^P \left[g_1(R_i) \right] = E^P[f(R_j) g_1(R_j)] =$$

$$= E^P[f(R_1) g_1(R_1)],$$

hence $R_1, ..., R_T$ are identically distributed under Q.

Above we used the property that random variables $\xi_1, ..., \xi_n$ are identically distributed under Q if and only if for any measurable and bounded function f we have $E^Q[f(\xi_j)] = E^Q[f(\xi_1)]$ for $j = 1, ..., n$. To prove this property it is enough to apply the equality $E^Q[f(\xi_j)] = E^Q[f(\xi_1)]$ to the functions $f(x) = f_y(x) := I_{x \le y}$.