Introduction to Mathematical Finance

Exercise sheet 8

Please submit your solutions online until Wednesday 10pm, 24/04/2024.

Exercise 8.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ and $Y = (Y_k)_{k\in\mathbb{N}_0}$ a supermartingale with respect to P and $\mathbb{F}$. Show that Y can be uniquely decomposed as $Y = Y_0 + M - A$, where M is a martingale with $M_0 = 0$ and A is predictable and increasing (i.e., $A_k \leq A_{k+1}$ P-a.s. for all k) with $A_0 = 0$. (This is the so-called *Doob decomposition* of Y.)

Exercise 8.2 Suppose that Y, $Z > 0$ and YZ are all martingales in discrete time. Give suitable additional assumptions under which

$$Y_. I_{\{\cdot \leq k\}} + \frac{Z_. Y_.}{Z_k} I_{\{\cdot > k\}}$$

is also a martingale for every k.

Exercise 8.3 Consider the one-step market with one risky asset S^1 and one riskless asset S^0, whose prices are given by

$$S_0^0 = 1, \qquad S_1^0 = 1 + r,$$

$$S_0^1 = 100, \qquad S_1^1 = 100(1 + \Delta X),$$

where $r > 0$ is a constant and $\Delta X \sim \mathcal{N}(\mu, \sigma^2)$. Consider the utility function

$$U(x) = \frac{1 - e^{-ax}}{a}, \quad a > 0.$$

Suppose that at time $t = 0$, we are given the amount of money A to invest in this market. Assume that there is no consumption. Find an optimal strategy $(A - \pi, \pi)$ which allocates the amount π to the risky asset and $A - \pi$ to the riskless asset, and maximizes the expected utility of the portfolio wealth.