Coordinator
Qinxin Yan

Introduction to Mathematical Finance

Exercise sheet 8

Please submit your solutions online until Wednesday 10pm, 24/04/2024.

Exercise 8.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ and $Y = (Y_k)_{k\in\mathbb{N}_0}$ a supermartingale with respect to P and $\mathbb{F}$. Show that Y can be uniquely decomposed as $Y = Y_0 + M - A$, where M is a martingale with $M_0 = 0$ and A is predictable and increasing (i.e., $A_k \leq A_{k+1}$ P-a.s. for all k) with $A_0 = 0$. (This is the so-called *Doob decomposition* of Y.)

Solution 8.1 If Y has a Doob decomposition as in above, then, since M is a martingale and A is predictable, we have

$$E[Y_k - Y_{k-1}|\mathcal{F}_{k-1}] = E[M_k - M_{k-1}|\mathcal{F}_{k-1}] - E[A_k - A_{k-1}|\mathcal{F}_{k-1}] = A_{k-1} - A_k.$$

In particular, since $A_0 = 0$, we have

$$A_k = -\sum_{j=1}^{k} E[Y_k - Y_{k-1}|\mathcal{F}_{k-1}].$$

So defining A as such yields the Doob decomposition of Y.

Exercise 8.2 Suppose that Y, $Z > 0$ and YZ are all martingales in discrete time. Give suitable additional assumptions under which

$$Y_\cdot I_{\{\cdot \leq k\}} + \frac{Z_\cdot Y_\cdot}{Z_k} I_{\{\cdot > k\}}$$

is also a martingale for every k.

Solution 8.2 First note that

$$Y_\cdot I_{\{\cdot \leq k\}} = Y_\cdot^k - Y_k I_{\{\cdot > k\}}.$$

By using this equality, we obtain

$$Y_\cdot I_{\{\cdot \leq k\}} + \frac{Z_\cdot Y_\cdot}{Z_k} I_{\{\cdot > k\}} = Y_\cdot^k + I_{\{\cdot > k\}} \frac{Z_\cdot Y_\cdot - Z_k Y_k}{Z_k} = U_\cdot^{(1)} + U_\cdot^{(2)},$$

where

$$U_\cdot^{(1)} := Y_\cdot^k, \quad U_\cdot^{(2)} := I_{\{\cdot > k\}} \frac{Z_\cdot Y_\cdot - Z_k Y_k}{Z_k}.$$

Since the first term $U^{(1)}$ is a stopped martingale, hence a martingale, we are done if the second term $U^{(2)}$ is also a martingale.

Assume that for all $\ell > k$, either the positive or the' negative part of $Z_\ell Y_\ell / Z_k$ is integrable. A consequence is that its integral and conditional expectations are well-defined (but possibly $\pm\infty$). Then, since the process $U_\cdot^{(2)}$ is identically zero until and including time k, we only need to verify the martingale condition for later time points. For $k \leq \ell \leq m$, we have

$$E\left[\frac{Z_m Y_m - Z_k Y_k}{Z_k}\bigg| \mathcal{F}_\ell\right] = \frac{E[Z_m Y_m | \mathcal{F}_\ell] - Z_k Y_k}{Z_k} = \frac{Z_\ell Y_\ell - Z_k Y_k}{Z_k},$$

so that $U_\cdot^{(2)}$ satisfies the martingale condition. Note that we are allowed to take out the $\mathcal{F}_l$-measurable factor $1/Z_k$ because $Z_k > 0$ and both $Z_m Y_m$ (by the martingale property of ZY) and the positive or negative part of $Z_m Y_m / Z_k$ (by our assumption) are integrable. We use the above equality in turn to show that $U^{(2)}$ is integrable. Indeed,

$$E\left[\frac{Z_\ell Y_\ell}{Z_k}\right] = E\left[E\left[\frac{Z_\ell Y_\ell}{Z_k}\bigg| \mathcal{F}_k\right]\right] = E\left[\frac{Z_k Y_k}{Z_k}\right] = E[Y_k] < \infty.$$

Exercise 8.3 Consider the one-step market with one risky asset S^1 and one riskless asset S^0, whose prices are given by

$$S_0^0 = 1, \qquad S_1^0 = 1 + r,$$

$$S_0^1 = 100, \qquad S_1^1 = 100(1 + \Delta X),$$

where $r > 0$ is a constant and $\Delta X \sim \mathcal{N}(\mu, \sigma^2)$. Consider the utility function

$$U(x) = \frac{1 - e^{-ax}}{a}, \quad a > 0.$$

Suppose that at time $t = 0$, we are given the amount of money A to invest in this market. Assume that there is no consumption. Find an optimal strategy $(A - \pi, \pi)$ which allocates the amount π to the risky asset and $A - \pi$ to the riskless asset, and maximizes the expected utility of the portfolio wealth.

Solution 8.3

Denote as the amount of money invested in the risky asset. Hence the amount of money invested in the riskless asset is $A - \pi$ and the portfolio wealth associated with this strategy is

$$V(\pi) = A + \pi\Delta X + (A - \pi)r \sim \mathcal{N}(\pi(\mu - r) + A(1 + r), \pi^2\sigma^2).$$

The expected utility of the portfolio is

$$E\Big[U\Big(V(\pi)\Big)\Big] = \frac{1}{a} - \frac{1}{a}E\Big[e^{-aV(\pi)}\Big];$$

hence in order to maximize the expected utility of the portfolio wealth, we need to find a strategy which minimizes the value $E\Big[e^{-aV(\pi)}\Big]$. Since

$$-aV(\pi) \sim \mathcal{N}\Big(-a(\pi(\mu - r) + A(1 + r)), a^2\pi^2\sigma^2\Big),$$

we get

$$E\Big[e^{-aV(\pi)}\Big] = -a(\pi(\mu - r) + A(1 + r)) + \frac{a^2\pi^2\sigma^2}{2},$$

which reaches its minimum at the point

$$\pi^* = \frac{\mu - r}{a\sigma^2}.$$

Thus, the optimal strategy is $(A - \pi^*, \pi^*)$.