Introduction to Mathematical Finance

Exercise sheet 9

Please submit your solutions online until Wednesday 10pm, 01/05/2024.

Exercise 9.1 Recall that an investment and consumption pair $(\psi, \tilde{c})$ with initial endowment $\tilde{v}_0$ is self-financing if $\psi_1 \cdot S_0 + \tilde{c}_0 = \tilde{v}_0$ and

$$\triangle\psi_{t+1} \cdot S_t + \tilde{c}_t = 0$$

for $t = 1, \dots, T-1$. Define the undiscounted wealth by $\tilde{W}_0 = \tilde{v}_0$ and $\tilde{W}_t := \psi_t \cdot S_t$ for $t = 1, \dots, T$, $W = \tilde{W}/S^0$ and $c = \tilde{c}/S^0$.

(a) Show in detail that $(\psi, \tilde{c})$ is self-financing if and only if

$$W_t = v_0 + \sum_{j=1}^{t} (\vartheta_j \cdot \triangle X_j - c_{j-1}) \quad \text{for } t = 0, \dots, T.$$

(b) Show that the pair $(\psi, \tilde{c})$ with initial wealth $\tilde{v}_0$ is self-financing if and only if

$$\tilde{W}_t = \tilde{v}_0 + \sum_{j=1}^{t} \left(\vartheta_j \cdot \triangle S_j - \tilde{c}_{j-1}\right) \quad \text{for } t = 0, ..., T.$$

Exercise 9.2 Recall that for each $t \in \{0, 1, ..., T\}$, suitable $\mathcal{F}_t$-measurable v_t and $(\vartheta', c') \in \mathcal{A}$, we define the *remaining conditional expected utility* to be

$$R_t(v_t, \vartheta', c') := E\left[\sum_{j=t}^{T} U_c(c'_j) + U_w\left(v_t + \sum_{j=t+1}^{T} (\vartheta'_j \cdot \triangle X_j - c'_{j-1}) - c'_T \right) \middle| \mathcal{F}_t \right].$$

Recall that

$$\mathcal{A}_t(\vartheta, c) := \{(\vartheta', c') \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j \leq t, c'_j = c_j \text{ for } j \leq t-1\}.$$

Show that for fixed $(\vartheta, c) \in \mathcal{A}$, we have

$$\operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}_t(\vartheta, c)} R_t(W_t^{v_0, \vartheta, c}, \vartheta', c') = \operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}} R_t(W_t^{v_0, \vartheta, c}, \vartheta', c').$$

Exercise 9.3

(a) For a twice differentiable utility function $U : (0, \infty) \to \mathbb{R}$, the so-called *absolute risk aversion* is given by

$$A(x) = -\frac{U''(x)}{U'(x)}.$$

Characterize all utility functions $U = U^a$ with constant absolute risk aversion $A(x) \equiv a > 0$. Normalize the functions so that $U^a(0) = 0$ and $(U^a)'(0) = 1$.

(b) Let $(\Omega, \mathcal{F}, P)$ be a general probability space. Assume the standard model on $(\Omega, \mathcal{F}, P)$. Suppose that U is strictly increasing. Show that if there is an arbitrage opportunity, then there is no solution to the utility maximisation problem

$$\max_{\vartheta \in \Theta} E\left[U(x + G_T(\vartheta))\right].$$

Exercise 9.4 Let (S^0, S^1) be an *arbitrage-free* financial market with time horizon T and assume that the bank account process $S^0 = (S^0_t)_{t=0,1,\dots,T}$ is given by $S^0_t = (1+r)^t$ for a constant $r \geq 0$. As usual, denote the set of all EMMs for S^1 with numeraire S_0 by $\mathbb{P}(S^0)$. Fix a $K > 0$. The undiscounted payoff of a *European call option* on S^1 with strike K and maturity $t \in \{1, \dots, T\}$ is denoted by C^E_t and given by

$$C^E_t = \left(S^1_t - K\right)^+,$$

whereas the undiscounted payoff of an *Asian call option* on S^1 with strike K and maturity $t \in \{1, \dots, T\}$ is denoted by C^A_t and given by

$$C^A_t := \left(\frac{1}{t}\sum_{j=1}^{t} S^1_j - K\right)^+.$$

(a) Fix a $\mathbb{Q} \in \mathbb{P}(S^0)$ and show that the function $\{1, \dots, T\} \to \mathbb{R}_+$, $t \mapsto E_{\mathbb{Q}}\left[\frac{C^E_t}{S^0_t}\right]$ is increasing.
Hint: Use Jensen's inequality for conditional expectations.

(b) Fix a $\mathbb{Q} \in \mathbb{P}(S^0)$ and show that for all $t = 1, \dots, T$, we have

$$E_{\mathbb{Q}}\left[\frac{C^A_t}{S^0_t}\right] \leq \frac{1}{t}\sum_{j=1}^{t} E_{\mathbb{Q}}\left[\frac{C^E_j}{S^0_j}\right]$$

(c) Fix a $\mathbb{Q} \in \mathbb{P}(S^0)$ and deduce that for all $t = 1, \dots, T$, we have

$$E_{\mathbb{Q}}\left[\frac{C^A_t}{S^0_t}\right] \leq E_{\mathbb{Q}}\left[\frac{C^E_t}{S^0_t}\right].$$

Interpret this inequality.