Introduction to Mathematical Finance

Exercise sheet 9

Please submit your solutions online until Wednesday 10pm, 01/05/2024.

Exercise 9.1 Recall that an investment and consumption pair $(\psi, \tilde{c})$ with initial endowment $\tilde{v}_0$ is self-financing if $\psi_1 \cdot S_0 + \tilde{c}_0 = \tilde{v}_0$ and

$$\triangle\psi_{t+1} \cdot S_t + \tilde{c}_t = 0$$

for $t = 1, \ldots, T-1$. Define the undiscounted wealth by $\tilde{W}_0 = \tilde{v}_0$ and $\tilde{W}_t := \psi_t \cdot S_t$ for $t = 1, \ldots, T$, $W = \tilde{W}/S^0$ and $c = \tilde{c}/S^0$.

(a) Show in detail that $(\psi, \tilde{c})$ is self-financing if and only if

$$W_t = v_0 + \sum_{j=1}^{t} (\vartheta_j \cdot \triangle X_j - c_{j-1}) \quad \text{for } t = 0, \ldots, T.$$

(b) Show that the pair $(\psi, \tilde{c})$ with initial wealth $\tilde{v}_0$ is self-financing if and only if

$$\tilde{W}_t = \tilde{v}_0 + \sum_{j=1}^{t} \left(\vartheta_j \cdot \triangle S_j - \tilde{c}_{j-1}\right) \quad \text{for } t = 0, ..., T.$$

Solution 9.1

(a) First discount the self-financing condition to get a condition in X, namely

$$\triangle\psi^0_{k+1} + \triangle\vartheta_{k+1} \cdot X_k + c_k = 0,$$
$$\psi^0_1 + \vartheta_1 \cdot X_0 + c_0 = v_0.$$

Using this,

$$\begin{aligned}
\triangle W_k &= \frac{\tilde{W}_k}{S^0_k} - \frac{\tilde{W}_{k-1}}{S^0_{k-1}} = \psi_k \cdot (1, X_k) - \psi_{k-1} \cdot (1, X_{k-1}) \\
&= \psi^0_k - \psi^0_{k-1} + \vartheta_k \cdot X_k - \vartheta_{k-1} \cdot X_{k-1} - \triangle\psi^0_k - \triangle\vartheta_k \cdot X_{k-1} - c_{k-1} \\
&= \vartheta_k \cdot \triangle X_k - c_{k-1}
\end{aligned}$$

for $k \geq 2$. Furthermore,

$$\triangle W_1 = \frac{\tilde{W}_1}{S^0_1} - v_0 = \psi^0_1 + \vartheta_1 \cdot X_1 - \psi^0_1 - \vartheta_1 \cdot X_0 - c_0 = \vartheta_1 \cdot X_1 - c_0.$$

Summing both results yields

$$W_k = v_0 + \sum_{j=1}^{k} (\vartheta_j \cdot \triangle X_j - c_{j-1}), \quad \text{for } k = 0, \ldots, T.$$

(b) We have

$$\triangle\tilde{W}_1 = \psi_1 \cdot S_1 - \tilde{v}_0 = \psi_1 \cdot S_1 - \psi_1 \cdot S_0 - \tilde{c}_0 = \psi_1 \cdot \triangle S_1 - \tilde{c}_0$$

as well as

$$\triangle\tilde{W}_k = \psi_k \cdot S_k - \psi_{k-1} \cdot S_{k-1} = \psi_k \cdot \triangle S_k + \triangle\psi_k \cdot S_{k-1} = \psi_k \cdot \triangle S_k - \tilde{c}_{k-1}$$

for all $k = 2, \dots, T$ if and only if $(\psi, \tilde{c})$ is self-financing. So we can sum up the increments to obtain

$$\tilde{W}_k = \tilde{W}_0 + \sum_{j=1}^{k} \triangle\tilde{W}_k = \tilde{v}_0 + \sum_{j=1}^{k} (\psi_k \cdot \triangle S_k - \tilde{c}_{k-1}).$$

Note that starting with these sums also gives the first two series of equalities, showing the equivalence.

Exercise 9.2 Recall that for each $t \in \{0, 1, ..., T\}$, suitable $\mathcal{F}_t$-measurable v_t and $(\vartheta', c') \in \mathcal{A}$, we define the *remaining conditional expected utility* to be

$$R_t(v_t, \vartheta', c') := E\left[\sum_{j=t}^{T} U_c(c'_j) + U_w\left(v_t + \sum_{j=t+1}^{T} (\vartheta'_j \cdot \triangle X_j - c'_{j-1}) - c'_T\right)\middle| \mathcal{F}_t\right].$$

Recall that

$$\mathcal{A}_t(\vartheta, c) := \{(\vartheta', c') \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j \leq t, c'_j = c_j \text{ for } j \leq t-1\}.$$

Show that for fixed $(\vartheta, c) \in \mathcal{A}$, we have

$$\operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}_t(\vartheta, c)} R_t(W_t^{v_0, \vartheta, c}, \vartheta', c') = \operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}} R_t(W_t^{v_0, \vartheta, c}, \vartheta', c').$$

Solution 9.2 Define

$$\mathcal{A}_k^{\text{post}}(\vartheta, c) := \{(\vartheta', c') \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j > k, c'_j = c_j \text{ for } j > k-1\}.$$

Note that $\mathcal{A} = \bigcup_{(\vartheta', c') \in \mathcal{A}} \mathcal{A}_k^{\text{post}}(\vartheta', c')$. For $(\vartheta', c') \neq (\vartheta'', c'')$, the sets $\mathcal{A}_k^{\text{post}}(\vartheta', c')$ and $\mathcal{A}_k^{\text{post}}(\vartheta'', c'')$ are either the same or disjoint. So for each distinct $\mathcal{A}_k^{\text{post}}(\vartheta', c')$, there exists $(\vartheta_0, c_0) \in \mathcal{A}_k(\vartheta, c)$ such that $\mathcal{A}_k^{\text{post}}(\vartheta', c') = \mathcal{A}_k^{\text{post}}(\vartheta_0, c_0)$. Hence,

$$\mathcal{A} = \bigcup_{(\vartheta', c') \in \mathcal{A}} \mathcal{A}_k^{\text{post}}(\vartheta', c') = \bigcup_{(\vartheta', c') \in \mathcal{A}_k(\vartheta, c)} \mathcal{A}_k^{\text{post}}(\vartheta', c').$$

Since $R_k(v_k, \vartheta', c')$ has the same value on each $\mathcal{A}_k^{\text{post}}(\vartheta', c')$, we have

$$\begin{aligned}\operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}} R_k(v_k, \vartheta', c') &= \operatorname{ess\,sup}\left\{R_k(v_k, \vartheta'', c'') : (\vartheta'', c'') \in \bigcup_{(\vartheta', c') \in \mathcal{A}_k(\vartheta, c)} \mathcal{A}_k^{\text{post}}(\vartheta', c')\right\} \\ &= \operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}_k(\vartheta, c)} R_k(v_k, \vartheta', c').\end{aligned}$$

This also holds for $v_k = W_k^{v_0, \vartheta, c}$.

Exercise 9.3

(a) For a twice differentiable utility function $U : (0, \infty) \to \mathbb{R}$, the so-called *absolute risk aversion* is given by

$$A(x) = -\frac{U''(x)}{U'(x)}.$$

Characterize all utility functions $U = U^a$ with constant absolute risk aversion $A(x) \equiv a > 0$. Normalize the functions so that $U^a(0) = 0$ and $(U^a)'(0) = 1$.

(b) Let $(\Omega, \mathcal{F}, P)$ be a general probability space. Assume the standard model on $(\Omega, \mathcal{F}, P)$. Suppose that U is strictly increasing. Show that if there is an arbitrage opportunity, then there is no solution to the utility maximisation problem

$$\max_{\vartheta \in \Theta} E\left[U(x + G_T(\vartheta))\right].$$

Solution 9.3

(a) Fix $a > 0$ and write $U := U^a$.

From the ODE

$$-\frac{U''(x)}{U'(x)} = a,$$

we get that

$$U(x) = C_1 e^{-ax} + C_2.$$

From $U'(0) = 1$, it follows that $C_1 = -1/a$, and from $U(0) = 0$, we get that $C_1 + C_2 = 0$, hence $C_2 = 1/a$. Thus, the normalized utility function with constant absolute risk aversion $a > 0$ is given by

$$U(x) = \frac{1 - e^{-ax}}{a}.$$

(b) Suppose that ϑ^* is an optimiser and ϑ^A is an arbitrage opportunity. Then

$$x + G_T(\vartheta^*) \leq x + G_T(\vartheta^* + \vartheta^A) \quad P\text{-a.s. with } P\left[G_T(\vartheta^*) < G_T(\vartheta^* + \vartheta^A)\right] > 0.$$

Set $\vartheta' := \vartheta^* + \vartheta^A$. Because U is increasing, $U(x + G_T(\vartheta')) \geq U(x + G_T(\vartheta^*))$ P-a.s., and because U is strictly increasing, also $P[x + G_T(\vartheta') > x + G_T(\vartheta^*)] > 0$. So $E[U(x + G_T(\vartheta'))] > E[U(x + G_T(\vartheta^*))]$ which contradicts the optimality of ϑ^*.

Exercise 9.4 Let (S^0, S^1) be an *arbitrage-free* financial market with time horizon T and assume that the bank account process $S^0 = (S^0_t)_{t=0,1,\dots,T}$ is given by $S^0_t = (1+r)^t$ for a constant $r \geq 0$. As usual, denote the set of all EMMs for S^1 with numeraire S_0 by $\mathbb{P}(S^0)$. Fix a $K > 0$. The undiscounted payoff of a *European call option* on S^1 with strike K and maturity $t \in \{1, \dots, T\}$ is denoted by C^E_t and given by

$$C^E_t = \left(S^1_t - K\right)^+,$$

whereas the undiscounted payoff of an *Asian call option* on S^1 with strike K and maturity $t \in \{1, \dots, T\}$ is denoted by C^A_t and given by

$$C^A_t := \left(\frac{1}{t}\sum_{j=1}^{t} S^1_j - K\right)^+.$$

(a) Fix a $\mathbb{Q} \in \mathbb{P}(S^0)$ and show that the function $\{1, \dots, T\} \to \mathbb{R}_+$, $t \mapsto E_{\mathbb{Q}}\left[\frac{C^E_t}{S^0_t}\right]$ is increasing.
Hint: Use Jensen's inequality for conditional expectations.

(b) Fix a $\mathbb{Q} \in \mathbb{P}(S^0)$ and show that for all $t = 1, \dots, T$, we have

$$E_{\mathbb{Q}}\left[\frac{C^A_t}{S^0_t}\right] \leq \frac{1}{t}\sum_{j=1}^{t} E_{\mathbb{Q}}\left[\frac{C^E_j}{S^0_j}\right]$$

(c) Fix a $\mathbb{Q} \in \mathbb{P}(S^0)$ and deduce that for all $t = 1, \dots, T$, we have

$$E_{\mathbb{Q}}\left[\frac{C^A_t}{S^0_t}\right] \leq E_{\mathbb{Q}}\left[\frac{C^E_t}{S^0_t}\right].$$

Interpret this inequality.

Solution 9.4

(a) It clearly suffices to show that for all $k = 1, \dots, T-1$, we have

$$E_Q\left[\frac{C^E_{k+1}}{S^0_{k+1}}\right] \geq E_Q\left[\frac{C^E_k}{S^0_k}\right]$$

Fix a $k \in \{1, \dots, T-1\}$. Using the *tower property* of conditional expectation, *Jensen's inequality* for conditional expectations (for the convex function $x \mapsto x^+$), the fact that S^1 is a Q-martingale and that $S^0_k = (1+r)$ is deterministic with $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{C^E_{k+1}}{S^0_{k+1}}\right] &= E_Q\left[\frac{\left(S^1_{k+1}-K\right)^+}{S^0_{k+1}}\right] \\
&= E_Q\left[\Big(\frac{S^1_{k+1}}{S^0_{k+1}} - \frac{K}{S^0_{k+1}}\Big)^+\right] \\
&= E_Q\left[E_Q\left[\Big(\frac{S^1_{k+1}}{S^0_{k+1}} - \frac{K}{S^0_{k+1}}\Big)^+\middle|\mathcal{F}_k\right]\right] \\
&\geq E_Q\left[E_Q\left(\left[\frac{S^1_{k+1}}{S^0_{k+1}} - \frac{K}{S^0_{k+1}}\middle|\mathcal{F}_k\right]\right)^+\right] \\
&= E_Q\left[\Big(\frac{S^1_{k}}{S^0_{k}} - \frac{K}{S^0_{k+1}}\Big)^+\right] \\
&= E_Q\left[\Big(\frac{S^1_{k}}{S^0_{k}} - \frac{K}{(1+r)S^0_{k}}\Big)^+\right] \\
&\geq E_Q\left[\Big(\frac{S^1_{k}}{S^0_{k}} - \frac{K}{S^0_{k}}\Big)^+\right] \\
&= E_Q\left[\frac{C^E_k}{S^0_k}\right]
\end{aligned}$$

(b) Since the function $x \mapsto x^+$ is convex, we have for $k = 1, \ldots, T$

$$\begin{aligned}
C^A_k = \left(\frac{1}{k}\sum_{j=1}^k S^1_j - K\right)^+ &= \left(\sum_{j=1}^k \frac{1}{k}\Big(S^1_j - K\Big)\right)^+ \\
&\leq \sum_{j=1}^k \frac{1}{k}\left(S^1_j - K\right)^+ = \frac{1}{k}\sum_{j=1}^k C^E_j.
\end{aligned}$$

By *linearity* and *monotonicity* of expectations and since $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{C^A_k}{S^0_k}\right] = E_Q\left[\frac{C^A_k}{(1+r)^k}\right] &\leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{(1+r)^k}\right] \\
&\leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{(1+r)^j}\right] = \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{S^0_j}\right]
\end{aligned}$$

(c) Putting the results of (a) and (b) together yields for $k = 1, \ldots, T$

$$E_Q\left[\frac{C^A_k}{S^0_k}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{S^0_j}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_k}{S^0_k}\right] = E_Q\left[\frac{C^E_k}{S^0_k}\right]$$

This means nothing else than that for a fixed EMM, the price of an Asian call option on S^1 is smaller than the price of the European call option on the same asset with the same strike price K and same maturity $k \in \{1, \ldots, T\}$.

This makes sense also intuitively since the price of a call option is increasing in the volatility of the underlying (because the probability of ending up in the money is higher), and averaging in the Asian call option amounts to reducing volatility of the underlying.