Mathematics for New Technologies in Finance

Exercise sheet 3

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J : \mathcal{C}_0^1(J, E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J, E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X) = (1, \mathbf{s}_1, \cdots, \mathbf{s}_M) \in \mathbf{T}^{(M)}(E)$. Let $X \in \mathcal{C}_0^1([0, s], E)$ and $Y \in \mathcal{C}_0^1([s, t], E)$.

Exercise 3.1 (Signatures)

(a) Let $X_t = t\mathbf{x} \in \mathbb{R}^d$ for all $t \in [0, 1]$. Calculate $\mathbf{Sig}_{[0,1]}(X)$.

(b) Let $X \in \mathcal{C}_0^1([0, T], E)$ and $X_0 = 0$. Prove that

$$\mathbf{Sig}_{[0,1]}(X)_{1,2} + \mathbf{Sig}_{[0,1]}(X)_{2,1} = \mathbf{Sig}_{[0,1]}(X)_1 \cdot \mathbf{Sig}_{[0,1]}(X)_2. \tag{1}$$

Exercise 3.2 (Calculate Signatures)

(a) Let $X \in \mathcal{C}_0^1([0, 1], \mathbb{R})$ s.t. $X_t = \sin(t)$ for all $t \in [0, 1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(b) Let $X \in \mathcal{C}_0^1([0, 1], \mathbb{R}^2)$ s.t. $X_t = (t, \sin(t))$ for all $t \in [0, 1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(c) Let $X \in \mathcal{C}_0^1([0, 1], \mathbb{R})$ and $n \in \mathbb{N}$. Calculate $\int_0^1 t^n dX_t$ when

  (i) $X_t = t$

  (ii) $X_t = \sin(t)$

(d) Prove that

$$\mathcal{F} = \left\{ \mathcal{C}_0^1([0, 1], \mathbb{R}) \ni X \mapsto \sum_{i=1}^{n} \lambda_i \int t^i dX_t \in \mathbb{R} : \forall \lambda_i \in \mathbb{R}, n \in \mathbb{N} \right\}$$

is a point-separating vector space. $\mathcal{C}_0^1([0, 1], \mathbb{R})$ is the space of all function f on $[0, 1]$ with $f(0) = 0$ and f has continuous derivative.

Exercise 3.3 (Controlled ODEs) Consider the controlled ODE: $X_0 = x \in \mathbb{R}$

$$dX_t^\theta = V^\theta(t, X_t^\theta)dt, \quad t \in [0, T]. \tag{2}$$

(a) Let

$$a_t = \frac{\partial X_T^\theta}{\partial X_t^\theta}. \tag{3}$$

Prove that

$$\frac{d}{dt} a_t = -\frac{\partial V^\theta}{\partial x}(t, X_t^\theta) \cdot a_t, \quad a_T = 1, \tag{4}$$

and relate a_t with $J_{t,T}$ in the lecture notebook.

(b) Prove that

$$\frac{d}{dt}\left(\frac{\partial X_t^\theta}{\partial \theta} a_t\right) = a_t \frac{\partial V^\theta}{\partial \theta}(t, X_t^\theta), \tag{5}$$

and

$$\frac{\partial X_T^\theta}{\partial \theta} = -\int_T^0 \frac{\partial X_T^\theta}{\partial X_t^\theta} \cdot \frac{\partial V^\theta}{\partial \theta}(t, X_t^\theta)dt. \tag{6}$$

(c) Is every feedforward neural network a discretization of controlled ODE?

Exercise 3.4 (Linear controlled ODE) Let $E = \mathbb{R}^d, W = \mathbb{R}^n$. Let $X \in \mathcal{C}_0^1([0,T], E)$ and let $B\colon E \to \mathbf{L}(W)$ be a bounded linear map. Consider

$$dY_t = B(dX_t)(Y_t) \tag{7}$$

If we denote $B^k = B(e_k)$, $k = 1, \cdots, d$ then

$$dY_t = \sum_{k=1}^{d} B^k(Y_t)dX_t^k. \tag{8}$$

Prove that

$$Y_t = \Big(\sum_{k=0}^{\infty} B^{\otimes k}\Big)\Big(\mathbf{Sig}_{[0,t]}(X)\Big)Y_0. \tag{9}$$

This implies that the solution of controlled SDE could be written as a linear function on signature stream of driving path. This implies that signature stream is a promising feature for controlled ODE.

References

[1] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv preprint arXiv:1603.03788*, 2016.

[2] Terry J Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths.* Springer, 2007.