Mathematics for New Technologies in Finance

Exercise sheet 4

Exercise 4.1 (Ito's formula) Let W be a Brownian motion on $[0,\infty)$ and define

$$Q^n(W)=\sum_{i=1}^{n}(W_{\frac{i}{n}}-W_{\frac{i-1}{n}})^2. \tag{1}$$

(a) Prove that $Q^n(W)$ converges to 1 in L^2

(b) Prove the following convergence in L^2 sense

$$\lim_{n\to\infty}\sum_{i=1}^{n}W_{\frac{i-1}{n}}(W_{\frac{i}{n}}-W_{\frac{i-1}{n}})=\frac{W_1^2-1}{2} \tag{2}$$

(c) Prove that if f is smooth and bounded

$$f(W_t)=f(0)+\int_0^t f'(W_s)dW_s+\int_0^t\frac{f''(W_s)}{2}ds. \tag{3}$$

Exercise 4.2 (Black-Scholes model) Let $\sigma>0$, $X_t=X_0\exp\{\sigma W_t-\frac{\sigma^2 t}{2}\}$.

(a) Prove that X is a solution of

$$dX_t=\sigma X_t dW_t.$$

(b) Let $K>0$, calculate

$$C_0=\mathbb{E}[(X_T-K)_+].$$

(c) Let $K>0$, calculate

$$\frac{\partial}{\partial X_0}\mathbb{E}[(X_T-K)_+].$$

Exercise 4.3 (Backpropagation) Translate a one layer neural network to a controlled ODE:

$$L^{(i)}: x\mapsto W^{(i)}x+a^{(0)}\mapsto\phi(W^{(1)}x+a^{(0)}),$$

with a cadlag control $u(t)=1_{[1,2)}(t)+2_{[2,\infty)}(t)$ and a time-dependent vector field

$$V(t,x)=1_{[0,1)}(t)\left(L^{(0)}(x)-x\right)+1_{[1,\infty)}(t)\left(L^{(1)}(x)-x\right).$$

The corresponding neural network at 'time' 3 is

$$x\mapsto L^{(0)}(x)\mapsto\phi(W^{(1)}L^{(0)}(x)+a^{(1)}).$$

(a) What is the evolution operator $J_{s,3}$?

(b) Calculate the derivative of the network with respect to parameters $W^{(1)}$ and $a^{(1)}$.