# Mathematics for New Technologies in Finance

## Exercise sheet 7

**Exercise 7.1 (Stochastic Descent)** Recall the calculation of Implied volatility using Bayes formula from Exercise sheet 6. Now we want to calculate the *implied volatility* $\sigma(K,T)$ from $C(K,T)$ using neural network. Proceed in the following steps:

- Define a neural network $f^\theta$ which takes as input the option price $C(K,T)$, the current price $S_0$, the strike price $K$, and the maturity $T$. The output will be the implied volatility $\sigma(K,T)$.
- Define a loss function $L$ which calculates the difference between the actual price $C(K,T)$ and $f^\theta(C(K,T),S_0,K,T)$ inserted in the Black-Scholes formula.
- Run a gradient descent.

**Exercise 7.2 (Breeden-Litzenberger formula)**

(a) Is there always a positive implied volatility $\sigma_{imp}$ related to the option price? If yes, prove it. Otherwise, on which price interval there is always a positive implied volatility $\sigma_{imp}$ related to the option price?

(b) Prove the Breeden-Litzenberger formula:

$$\partial_K^2 C(T,K)dK = \mathrm{law}(S_T)(dK). \tag{1}$$

(c) Discretize the Breeden-Litzenberger formula and link it with Butterfly spreads.

**Exercise 7.3 (Dupire formula)** Assume the following local volatility model:

$$dS_t = \sigma(t,S_t)S_t dW_t. \tag{2}$$

(a) If $\sigma(t,S_t) = \sigma S_t^\beta$, for which value of $\beta$, the market has leverage effect (the volatility increases when the stock price goes down), which is empirically observed.

(b) Let $V_t$ be the fair price of an European payoff $h(S_T)$. Prove the backward Kolmogorov equation:

$$\partial_t V_t + \frac{1}{2}\sigma(S,t)^2 S^2 \partial_{SS}^2 V_t = 0 \tag{3}$$

(c) Let $f_T^S$ be the probability density function of $S_T$, prove the forward Kolmogorov equation (Fokker-Planck equation):

$$\partial_T f(S,T) = \frac{1}{2}\partial_S^2\Big(\sigma(S,T)^2 S^2 f(S,T)\Big) \tag{4}$$

(d) Prove by Fokker-Planck equation the Dupire formula:

$$\sigma^2(K,T) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2\partial_K^2 C(T,K)} \tag{5}$$

where $C(T,K)$ is the European call option price of maturity $T$ and strike $K$.

## References

[1] Pierre Henry-Labordère. Calibration of local stochastic volatility models to market smiles: A monte-carlo approach. *Risk Magazine, September*, 2009.