Mathematics for New Technologies in Finance

Solution sheet 4

Exercise 4.1 (Ito's formula) Let W be a Brownian motion on $[0,\infty)$ and define

$$Q^n(W) = \sum_{i=1}^{n} (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2. \tag{1}$$

(a) Prove that $Q^n(W)$ converges to 1 in L^2

(b) Prove the following convergence in L^2 sense

$$\lim_{n\to\infty} \sum_{i=1}^{n} W_{\frac{i-1}{n}}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) = \frac{W_1^2 - 1}{2} \tag{2}$$

(c) Prove that if f is smooth and bounded

$$f(W_t) = f(0) + \int_0^t f'(W_s)dW_s + \int_0^t \frac{f''(W_s)}{2}ds. \tag{3}$$

Solution 4.1

(a)

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{i=1}^{n}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2 - 1\Big)^2\Big] &= \mathrm{Var}(\sum_{i=1}^{n}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2) \\
&= \sum_{i=1}^{n} \mathrm{Var}((W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2) \\
&= n(\mathbb{E}(W^4_{\frac{1}{n}}) - \frac{1}{n^2}) \\
&= n(\frac{3}{n^2} - \frac{1}{n^2}) \to 1 \quad \text{as } n \to \infty
\end{aligned}$$

(b) Combining (a) and the fact that

$$2W_{\frac{i-1}{n}}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) = (W_{\frac{i}{n}} + W_{\frac{i-1}{n}})(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) - (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2.$$

(c) By Ito's formula

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$

Then taking integral of both sides gives us the result.

Exercise 4.2 (Black-Scholes model) Let $\sigma > 0$, $X_t = X_0 \exp\{\sigma W_t - \frac{\sigma^2 t}{2}\}$.

(a) Prove that X is a solution of

$$dX_t = \sigma X_t dW_t.$$

(b) Let $K > 0$, calculate

$$C_0 = \mathbb{E}[(X_T - K)_+].$$

(c) Let $K > 0$, calculate

$$\frac{\partial}{\partial X_0}\mathbb{E}[(X_T - K)_+].$$

Solution 4.2

(a)

$$\begin{aligned} dX_t &= d[X_0 \exp(\sigma W_t - \frac{\sigma^2 t}{2})] = X_0 d[\exp(\sigma W_t - \frac{\sigma^2 t}{2})] \\ &= X_0 \exp(\sigma W_t - \frac{\sigma^2 t}{2})(\sigma dW_t + \frac{1}{2}\partial_t^2 W_t dt - \frac{\sigma^2 dt}{2}) \\ &= \sigma X_t dW_t \end{aligned}$$

(b) Applying Black-Scholes formula, we have

$$C_0 = X_0\Phi(d_1) - K\Phi(d_2) \tag{4}$$

where

$$d_1 = \frac{\log(\frac{X_0}{K}) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

(c)

$$\frac{\partial}{\partial X_0}\mathbb{E}[(X_T - K)_+] = \frac{\partial}{\partial X_0}(X_0\Phi(d_1) - K\Phi(d_2)) = \phi(d_1)$$

Exercise 4.3 (Backpropagation) Translate a one layer neural network to a controlled ODE:

$$L^{(i)} : x \mapsto W^{(i)}x + a^{(0)} \mapsto \phi(W^{(1)}x + a^{(0)}),$$

with a cadlag control $u(t) = 1_{[1,2)}(t) + 2_{[2,\infty)}(t)$ and a time-dependent vector field

$$V(t,x) = 1_{[0,1)}(t)\left(L^{(0)}(x) - x\right) + 1_{[1,\infty)}(t)\left(L^{(1)}(x) - x\right).$$

The corresponding neural network at 'time' 3 is

$$x \mapsto L^{(0)}(x) \mapsto \phi(W^{(1)}L^{(0)}(x) + a^{(1)}).$$

(a) What is the evolution operator $J_{s,3}$?

(b) Calculate the derivative of the network with respect to parameters $W^{(1)}$ and $a^{(1)}$.

Solution 4.3

(a) $J_{s,3}v = v + 1_{[0,1[}(s)\left(dL^{(1)}\left(X_{s-}\right)dL^{(0)}(x)v - v\right) + 1_{[1,2[}(s)\left(dL^{(1)}\left(X_{s-}\right)v - v\right)$

(b)

$$\frac{\partial V}{\partial W^{(1)}} = 1_{[0,1)}(t)\frac{\partial L^{(0)}}{\partial W^{(1)}} + 1_{[1,\infty)}(t)\frac{\partial L^{(1)}}{\partial W^{(1)}} = 1_{[1,\infty)}(t)\frac{\partial L^{(1)}}{\partial W^{(1)}} = 1_{[1,\infty)}(t)W^{(1)}\frac{\partial}{\partial W^{(1)}}\phi(W^{(1)}x + a^{(1)}).$$

Similarly,

$$\frac{\partial V}{\partial a^{(1)}} = 1_{[1,\infty)}(t)W^{(1)}\frac{\partial}{\partial a^{(1)}}\phi(W^{(1)}x + a^{(1)}).$$

Since we have

$$\partial X_T^\theta = \sum_{i=1}^d \int_0^T J_{s+,T}\partial V_i^\theta\left(s-, X_{s-}^\theta\right)du^i(s)$$

We have

$$\partial X_3^{W^{(1)}} = \sum_{i=1}^d \int_1^3 J_{s+,3}\frac{\partial V}{\partial W^{(1)}}du^i(s) \text{ and } \partial X_3^{a^{(1)}} = \sum_{i=1}^d \int_1^3 J_{s+,3}\frac{\partial V}{\partial a^{(1)}}du^i(s)$$