Mathematics for New Technologies in Finance

Solution sheet 8

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J : \mathcal{C}_0^1(J, E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J, E)$.

Exercise 8.1 (Signatures and reservoirs computing)

(a) Let $X \in \mathcal{C}_0^1([0, T], \mathbb{R}^n)$ satisfying the dynamic:

$$dX_t = \sum_{k=1}^{m} V_k(X_t) dW_t^k, \quad X_t \in \mathbb{R}^n, W_t \in \mathbb{R}^m, V^k : \mathbb{R}^n \to \mathbb{R}^n, \tag{1}$$

where $(W_t)_{t=0}^{\infty}$ is a Brownian motion. Prove that

$$X_t = \sum_{d=0}^{\infty} \sum_{i_1, \cdots, i_d=1}^{n} \left(\int_{0 \le t_1 \le \cdots \le t_d \le t} dW_{t_1}^{i_1} \cdots dW_{t_d}^{i_d} \right) V^{i_d} \cdots V^{i_1}(X_0) \cdot X_0. \tag{2}$$

where

$$Vf(x) = df(x) \cdot V(x).$$

(b) Rewrite (2) with signature in the form of the following:

$$X_t = \langle \mathbf{R}, \mathbf{Sig}_{[0,t]}(W) \rangle X_0, \tag{3}$$

and express the readout $\mathbf{R}$ with $(V^k)_{k=1}^m$ (notice that $\mathbf{R}$ depends on X_0).

(c) Relate (3) with reservoirs computing.

Solution 8.1

(a)

$$X_t = X_0 + \sum_{i=0}^{d} \int_0^t V_i(X_s) dW^i(s) \tag{4}$$

By iterating using Picard iteration we can get

$$X_t = \sum_{d=0}^{\infty} \sum_{i_1, \cdots, i_d=1}^{n} \left(\int_{0 \le t_1 \le \cdots \le t_d \le t} dW_{t_1}^{i_1} \cdots dW_{t_d}^{i_d} \right) V^{i_d} \cdots V^{i_1} (X_0) \cdot X_0 \tag{5}$$

(b) The Signature of a bounded variation path W over the interval $[0, t]$ is the tensor sequence

$$\mathbf{Sig}_{[0,t]}(W) := \int_{0 \le t_1 \le \cdots \le t_d \le t} dW_{t_1}^{i_1} \cdots dW_{t_d}^{i_d} \tag{6}$$

Then we can express

$$X_t = \sum_{d=0}^{\infty} \sum_{i_1, \cdots, i_d=1}^{n} \mathbf{Sig}_{[0,t]}(W) V^{i_d} \ldots V^{i_1} (X_0) \cdot X_0 \tag{7}$$

(c) This explains that any solution can be represented, up to a linear readout, by a universal reservoir, namely signature.

Exercise 8.2 In Exercise 8.1, denote the dimension of state space by N and the number of Brownian motion by d

(a) Choose a dimension N and some random matrices $A_1, \dots, A_d$, then consider the Taylor expansion. Look at the solution of this system and describe their relation to signature. Can we express all signature components from this system?

(b) Choose a dimension N and some random NN vector fields of type $\sigma(A_1. + b_1), \dots, \sigma(A_d. + b_d)$. Consider a learning task $u \mapsto \sup(u(.))$ and solve the regression problem on path space (Lipschitz functions) by a regression.

Solution 8.2

(a) Let $V_i(X) = A_i X$, $X_t \in \mathbb{R}^n$, we have the new dynamic:

$$dZ_t = \sum_{i=1}^{d} A_i Z_t dW^i(t) \tag{8}$$

Then we have the solution

$$Z_t = \sum_{d=0}^{\infty} \sum_{i_1, \cdots, i_d = 1}^{n} \left(\int_{0 \le t_1 \le \cdots \le t_d \le t} dW_{t_1}^{i_1} \cdots dW_{t_d}^{i_d} \right) A_{i_d} \cdots A_{i_1} \cdot Z_0 \tag{9}$$

We cannot express all signature components from this system.

(b) Similarly, the new dynamic becomes:

$$dZ_t = \sum_{i=1}^{d} \sigma(A_i Z_t + b_i) dW^i(t) \tag{10}$$

We want to minimize the loss function

$$L = \frac{1}{N} \sum_{1}^{N} l(X_t, Z_t) \tag{11}$$

References

[1] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv preprint arXiv:1603.03788*, 2016.

[2] Terry J Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Springer, 2007.