Igor Kortchemski Condensation phenomen Chapter 3: Application to roudom trees to rendom trees ETH, Spring 2024 Outline : 1) Coding Bienaymé træs 2) The cycle lenne and the Vervaat transform 3) Condensation in subvitical træs In short, the goal is to identify a condensation phenomenon / one big degree phenomenon in large substituel Bienayme trees with heavy-tailed offspring distribution. 1) Coding Bienayme Arees a) Trees Here we work with plane trees (sometimes called noted ordered trees), for example: $T_1 = 2 \phi_{1,1}, z, 21, 22 ;$ $T_2 = 5 \phi_{1,2}, z, 11, 12 ;$ T2 = 2, \$ 1, 2, 11, 12 Z

Formally, they are defined as actain sets of habels (sequences of integers)
Definition Set
$$U = \bigcup_{N \gg 0}^{n} With N = \Sigma(1, 7, ..., 3)$$
 and $N = \Sigma \neq \Sigma$.
A plane knee T is a finite subset of U (called vertices) such that
(1) $\not= \Sigma = (1, 2, ..., 3)$ and $N = \Sigma \neq \Sigma$.
(2) if $(V_{11}, ..., V_{n}) \in T$, then $(V_{11}, ..., V_{n-1}) \in T$ (called the parent of $(V_{11}, ..., V_{n})$)
(3) $T \in V = (V_{11}, ..., V_{n}) \in T$, there is an integer $k_{V}(T) \ge 0$ such that $(V_{11}, ..., V_{n}) \in T$ if $i \le k_{V}(T)$
(called the vumber of children of V , or a bit abarively degree of V)
Ne denote by $|T|$ the same of T (its number of vertices)

Informally, a plane tree can be seen as a generalogical tree where individuals are the vertices

Defenition Let T be a tree with size n, with vertices ordered in lexicographical order: Uo CU, C. -- CU, The dubasieurs path 25 (T)= (20, CT1, --; 25, (T)) is defined by:

- · W_(T)=0
- $\mathcal{W}_{i\pi}(\tau) = \mathcal{W}_{i}(\tau) + k_{\mathcal{U}_{i}}(\tau) i \int_{\mathcal{V}} osis |\tau| i.$



6) Bienaymé Arees

Bienaqué trees are, roughly speaking, roudon trees which describe the genealogical tree of a population where individuals have a roudon number of children, Endependently, destributed according to a scane probability distribution, called the offspring distribution.

Such models were couridered by Bienayné (1845) and Falton & Watson (1875) who were interested in estimating the probability of extinction of noble names.

In
$$\mu = (\mu(1); 1>0)$$
 be a probability distribution on Z₁, called the officing distribution.
We always assume that $\mu(0) + \mu(1) < 1$.
Let T to the set of all first tree (T = contable)
Theorem For a finite tree T = T, we define.
B₁ (T) = T = (1 (k₁(T)) CV)
II $\sum_{i=1}^{n} \lambda_{i}(0) \leq 1$ then C₁ is a probability measure on T
This is of cause connected to the fad that a probability measure on T
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Start: By biompaning according to the number of diction of the root, we have:
 $c = \sum_{i=1}^{n} \sum_{j=1}^{n} E_{\mu}(i) = \sum_{i=1}^{n} h_{i}(i) = \mu(i) = \frac{1}{k_{2}} h_{i}(i) =$

Remark When
$$\sum_{i=1}^{n} (i_{i}) > 1$$
, it is possible to define a perturbed to use the set of all plane (not reasonable finite) traces such that (20) holds for finite trace (3), is the bard of (1) in the previous passif).
In the sequel we always obtaine that $\sum_{i=1}^{n} (\mu(i)) = 1$
A By random free with $k = T - reduced random variable, with law By.
To make the convection with the Lithericanse path, we introduce the random welk
(Whiles) to (X_{1,22}) be not random variable with law By.
To make the convection with the Lithericanse path, we introduce the random welk
(Whiles) to (X_{1,22}) be not random variable with law B(X_2=k)=p(kn) for 63.5.
Set we =0 and Win=X_1-+X_n for no.4.
Also set $S=inf(X,k_{21})$ be no surder there. Then
(21(Y),..., Wing (1))!!" (We,..., WE)
The people is straightfound uning (2) by computing the peopleticity that the 2 readom vector are
equal to (m_1..., w.).
In the sequel, Yn denotes a By, readom true conditioned on having in vectors (we
implicitly reative to values of n such that B((17)=n)>0).
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(see the first by the set that this (conditionary is "non local." To order if "local" we
we are going to me the so-called right laws.
Evel of local states by local set (17)$

c) (ritical Bienayme trees

Observe that $E[X_1] = \sum_{k \ge -1} k \mu(k+1) = \sum_{k \ge 1} k \mu(k) - 2$ In particular, critical frees (for which $\sum_{k \ge 1} k \mu(k) = 1$) play a special role: we have $E[X_1] = 2$ iff μ is vitical.

There is an important took which allows to "true" the mean:

$$\frac{P_{500}}{P_{\mu}(T)} = \frac{1}{T} \hat{\mu} (k_{\mu}(T)) = \frac{1}{TT} \int_{u \in T} \frac{b^{\mu}(T)}{p_{\mu}(b)} + (k_{\mu}(T)) = \frac{1}{G_{\mu}(b)^{\mu}} \frac{b^{-1}}{B_{\mu}(T)} \frac{B_{\mu}(T)}{B_{\mu}(T)}$$
Suming on TET we get $B_{\mu}(T_{n}) = \frac{1}{G_{\mu}(b)} \frac{b^{-1}}{B_{\mu}(T_{n})}$, so that $\frac{\hat{\mu}(T)}{B_{\mu}(T_{n})} = \frac{B_{\mu}(T)}{B_{\mu}(T_{n})}$.

Remark In the previous notation,
$$\hat{\mu}$$
 has mean $\frac{bE_{\mu}(b)}{C_{\mu}(b)}$. Thus μ can be exponentially tilled to a critical offering distribution is $\frac{2F_{\mu}(a)}{2\pi R} > 1$, where R is the radius of convergence of R (one can show that $x \mapsto \frac{2F_{\mu}(a)}{2\pi R} > 1$, where R is the radius of convergence of R (one can show that $x \mapsto \frac{2F_{\mu}(a)}{2\pi R} > 1$).
Observe that f is always possible in the following two cases:
• subcritical offering distributions with $R = \infty$ (exercise)
Not clivarys possible (take a subcritical offering distribution with $R = 1$)

We first introduce some notation. Set $S_n = \sum (\alpha_{1,\dots,\alpha_n}) \in \{-1,0,1,\dots,3^n\}$: $\alpha_i + \dots + \alpha_n = -i \sum$. Recell that $\overline{S}_n = \sum (\alpha_{1,\dots,\alpha_n}) \in \{-1,0,1,\dots,3^n\}$: $\alpha_i + \dots + \alpha_n = -i$ and $\alpha_1 + \dots + \alpha_i > -i$ for $1 \le i \le n \ge 1$. We identify $\mathbb{Z}'_n \mathbb{Z}$ with $\sum 0, 1, \dots, n-i \ge 1$. For $\alpha = (\alpha_{1,\dots,\alpha_n}) \in \mathbb{B}^n$ and $i \in \mathbb{Z}/n\mathbb{Z}$, we define $\alpha_{i+1}^{(i)} = (\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_{i+n})$ with addition considered module n.



Theorem (cycle laune) For every $x \in S_n$, set $I(x) = \xi i \in \mathbb{Z}/n\mathbb{Z} : x^{(i)} \in S_n \xi$. Then $(\operatorname{Cerl}(I(x)) = 1$ and the element of I(x) is the first time when the walk with jumps given by x reaches its infimum for the first time

In the provises example $I(x) = \xi \exists \tilde{\varsigma}$. The proof is not complicated but a bit tedious to write: it is left to the reader. We now see some probabilistic consequences

6) Probabilistic consequences

Definition A function F: Z -> R is invariant under cyclic permutation if fixe Z, fiezz/nZ, F(z)=F(z⁽ⁱ⁾) Exouples x, + + xn, x, xn, mox (x1,...,xn) Recall then $W_n = X_1 + \dots + X_n$ is the random walk with $D(X_1 = k) = \mu(k+1)$ for $k \ge -1$ and $S = irg \{ k \ge 1 : W_R = -1 \}$ Proposition Let F: Z > B be invariant under cyclic shifts with F>0 or F(X1,...,Xn) EL2. 1) $\mathbb{E}[F(X_{1,...}, X_{n})] = \frac{1}{n} \mathbb{E}[F(X_{1,...}, X_{n})]$ 2) $\mathcal{B}(3=n) = \frac{1}{n} \mathcal{B}(W_n = -1)$ 3) The hour of F(X1,..., Xn) under B(·15=n) is equal to the law of F(X1,..., Xn) under B(·1W1=n) Before the most, we give an application for Bienagné trees: Application Assume that mis which, has finite variance with apprivation rupport. let I'm be Bu-tree conditioned on having a vertices. Then $\frac{1}{\sqrt{n}}$ max $k_u(T_n) \xrightarrow{B} O$. Proof of the explication mex Ru(Mn)+1 is the mersional jump of W(Mn), which has the same law $e_{\mathcal{S}}(W_{\mathcal{R}}: \mathfrak{d} \leq k \leq n)$ under $\mathcal{B}(\cdot | \zeta = n)$ Thus nex $k_n(T_n)$ $\stackrel{(d)}{=}$ nex X_i under $\mathcal{B}(\cdot|J=n) \stackrel{(d)}{=}$ nex X_i ner n_n n_{SiSN} under B (· [Wn=-1] But X1 is centered, has finite variance and is aperiodic. Thus by exercise 4: mer Xi BO render B(. [Wn=-1]). (Formally it is with Wn=0 bet the proof is exactly Vit

Pressed of the propertition First observe that once i) is shown:
• 2) follows by taking F=1
• 3) follows by dividing the equality of a) by the equality of 2)
For a) we introduce the volution
$$\vec{X}_n = (X_{11}...,X_n)$$
. Then $\{S = n\} = \{\vec{X}_n \in S_n\}$, $\{W_n = n\} = \{\vec{X}_n \in S_n\}$
eand $\forall i \in \mathbb{Z}^n \mathbb{Z}, \ \vec{X}_n^{(i)} \stackrel{(i)}{=} \vec{X}_n$. Then write
 $\mathbb{E}[F(\vec{X}_n) \land \vec{X}_n^{(i)}] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[F(\vec{X}_n^{(i)}) \land \vec{X}_n^{(i)} \in S_n]$
 $= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[F(\vec{X}_n) \land \vec{X}_n^{(i)} \in S_n]$
 $= \frac{1}{n} \mathbb{E}[F(\vec{X}_n) \stackrel{(i)}{=} \vec{X}_n \quad \text{Then write}$
 $= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[F(\vec{X}_n) \land \vec{X}_n^{(i)} \in S_n]$
 $= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[F(\vec{X}_n) \stackrel{(i)}{=} \vec{X}_n^{(i)} \in S_n]$
 $= \frac{1}{n} \mathbb{E}[F(\vec{X}_n) \stackrel{(i)}{=} \vec{X}_n^{(i)} \in S_n]$
 $= \frac{1}{n} \mathbb{E}[F(\vec{X}_n) \stackrel{(i)}{=} \vec{X}_n^{(i)} \in S_n]$ by Furbini
 $= 1_{\vec{X}_n \in S_n} (\text{cyclic lowne})$
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