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Condensation phenomena in random trees
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Chapter 3: Application to random trees

Outline: 1) Coding Bienaymé trees
2) The cycle lemma and the Vervaat transform
3) Condensation in subcritical trees

In short, the goal is to identify a condensation phenomenon / one big degree phenomenon in large subcritical Bienaymé trees with heavy-tailed offspring distribution.

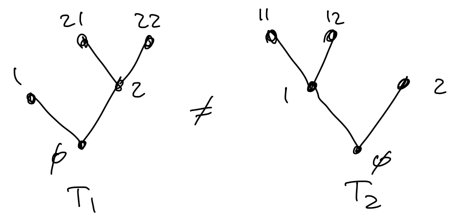
1) Coding Bienaymé trees

a) Trees

Here we work with plane trees (sometimes called rooted ordered trees), for example:

$T_1 = \{\phi, 1, 2, 21, 22\}$

$T_2 = \{\phi, 1, 2, 11, 12\}$



Formally, they are defined as certain sets of labels (sequences of integers)

Definition Set $\mathcal{U} = \bigcup_{n \geq 0} \mathbb{N}^n$ with $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}^0 = \{\phi\}$.

A plane tree T is a finite subset of $\mathcal{U}$ (called vertices) such that

(1) $\phi \in T$ (called the root)

(2) if $(v_1, \ldots, v_n) \in T$, then $(v_1, \ldots, v_{n-1}) \in T$ (called the parent of $(v_1, \ldots, v_n)$)

(3) If $v = (v_1, \ldots, v_n) \in T$, there is an integer $k_v(T) \geq 0$ such that $(v_1, \ldots, v_n, i) \in T$ iff $i \leq k_v(T)$ (called the number of children of v, or a bit abusively degree of v)

We denote by $|T|$ the size of T (its number of vertices)

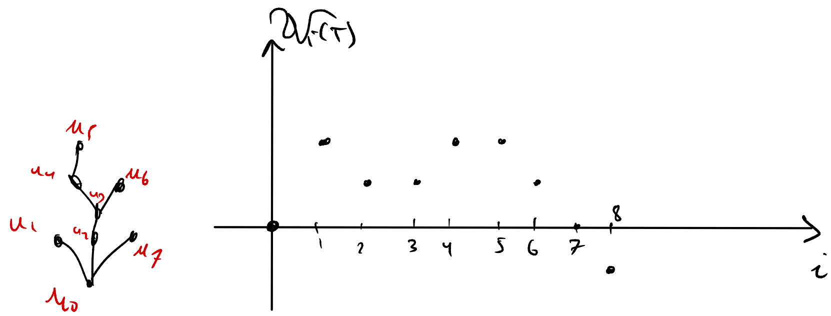
Informally, a plane tree can be seen as a genealogical tree where individuals are the vertices

Definition The lexicographical order on $\mathcal{U}$ is defined as follows: $v<w$ if there exists $u\in\mathcal{U}$ such that $v=u(v_1,\dots,v_n)$, $w=u(w_1,\dots,w_n)$ and $v_1<w_1$.

Definition Let T be a tree with size n, with vertices ordered in lexicographical order: $u_0<u_1<\dots<u_{n-1}$. The Łukasiewicz path $\mathcal{W}(T)=(\mathcal{W}_0(T),\dots,\mathcal{W}_n(T))$ is defined by:

- $\mathcal{W}_0(T)=0$
- $\mathcal{W}_{i+1}(T)=\mathcal{W}_i(T)+k_{u_i}(T)-1$ for $0\le i\le |T|-1$.

Example



Proposition The map

$$\begin{aligned}\{\text{trees with } n \text{ vertices}\} &\longrightarrow \bar{S}_n\\ T &\longmapsto (k_{u_i}(T)-1 : 0\le i\le n-1)\end{aligned}$$

is a bijection, where $\bar{S}_n=\{(x_1,\dots,x_n)\in\{-1,0,1,\dots\}^n : x_1+\dots+x_n=-1,\ x_1+\dots+x_i>-1 \text{ for } 1\le i\le n\}$

This can be readily shown by induction. The complete proof is a bit tedious to write and it is skipped here (the reader should convince him/her that this is true)

b) Bienaymé trees

Bienaymé trees are, roughly speaking, random trees which describe the genealogical tree of a population where individuals have a random number of children, independently, distributed according to a same probability distribution, called the offspring distribution.

Such models were considered by Bienaymé (1845) and Galton & Watson (1875) who were interested in estimating the probability of extinction of noble names.

Let $\mu = (\mu(i) : i \geq 0)$ be a probability distribution on $\mathbb{Z}_+$, called the offspring distribution.
We always assume that $\mu(0) + \mu(1) < 1$.
Let $\mathbb{T}$ be the set of all finite trees ($\mathbb{T}$ is countable)

Theorem For a finite tree $T \in \mathbb{T}$, we define
$$\mathbb{P}_\mu(T) = \prod_{u \in T} \mu(k_u(T)) \quad (\forall)$$
If $\sum_{i=1}^{\infty} i\mu(i) \leq 1$ then $\mathbb{P}_\mu$ is a probability measure on $\mathbb{T}$

This is of course connected to the fact that a μ-branching process dies out a.s iff $\sum_{i=0}^{\infty} i\mu(i) \leq 1$
(Bienaymé 1845, Galton-Watson 1875, Steffensen 1930).

Proof Set $c = \sum_{t \in A} \mathbb{P}_\mu(t)$. We show that $c = 1$.

Step 1: By decomposing according to the number of children of the root, we have:
$$c = \sum_{k \geq 0} \sum_{\substack{t \in \mathbb{T} \\ k_\emptyset(t) = k}} \mathbb{P}_\mu(t) = \sum_{k \geq 0} \sum_{t_1, \dots, t_k \in \mathbb{T}} \mu(k)\mathbb{P}_\mu(t_1) \cdots \mathbb{P}_\mu(t_k) = \sum_{k \geq 0} \mu(k) c^k$$

Step 2 Set $f(s) = \sum_{k=0}^{\infty} \mu(k) s^k - s$ then $f(0) = \mu(0) > 0$, $f(1) = 0$, $f'(1) \leq 0$ and $f'' > 0$ on $[0,1]$
Thus the only solution of $f(s) = 0$ on $[0,1]$ is $s = 1$.

Step 3 We check that $c \leq 1$ by constructing a random variable whose "law" is $\mathbb{P}_\mu$. To do this, let $(K_u)_{u \in \mathcal{U}}$ be iid random variables with law μ. Set
$$\mathcal{T} = \{(u_1, \dots, u_n) \in \mathcal{U} : u_i \leq K_{(u_1, \dots, u_{i-1})} \text{ for every } 1 \leq i \leq n\}$$
(intuitively, K_u is the number of children of $u \in \mathcal{U}$, if u is in the tree). Observe that $\mathcal{T}$ is possibly infinite. But for $t \in \mathbb{T}$, we have
$$\mathbb{P}(\mathcal{T} = t) = \mathbb{P}(K_u = k_u(t) \text{ for every } u \in t) = \prod_{u \in t} \mu(k_u(t)) = \mathbb{P}_\mu(t)$$
Thus $c = \sum_{t \in \mathbb{T}} \mathbb{P}_\mu(t) = \sum_{t \in \mathbb{T}} \mathbb{P}(\mathcal{T} = t) = \mathbb{P}(\mathcal{T} \in \mathbb{T}) \leq 1$

By the first two steps we conclude that $c = 1$.

Remark When $\sum_{i=0}^{\infty} i\mu(i) > 1$, it is possible to define a probability measure $\mathbb{P}_\mu$ on the set of all plane (not necessarily finite) trees such that $(*)$ holds for finite trees ($\mathbb{P}_\mu$ is the law of τ in the previous proof).

In the sequel we always assume that $\sum_{i=0}^{\infty} i\mu(i) \le 1$

A B_μ random tree will be a $\mathbb{T}$-valued random variable, with law $\mathbb{P}_\mu$.

To make the connection with the Lukasiewicz path, we introduce the random walk $(W_n)_{n\ge 0}$: let $(X_i)_{i\ge 1}$ be iid random variables with law $\mathbb{P}(X_1 = k) = \mu(k+1)$ for $k \ge -1$.
Set $W_0 = 0$ and $W_n = X_1 + \dots + X_n$ for $n \ge 1$.
Also set $\zeta = \inf\{k \ge 1 : W_k = -1\} \in \mathbb{N} \cup \{+\infty\}$

Proposition Let τ be a B_μ random tree. Then
$$(\mathcal{W}_0(\tau), \dots, \mathcal{W}_{|\tau|}(\tau)) \overset{law}{=} (W_0, \dots, W_\zeta)$$

The proof is straightforward using $(*)$ by computing the probability that the 2 random vectors are equal to $(w_0, \dots, w_n)$.

In the sequel, τ_n denotes a B_μ random tree conditioned on having n vertices (we implicitly restrict to values of n such that $\mathbb{P}(|\tau| = n) > 0$).

Corollary
- $|\tau| \overset{law}{=} \zeta$
- $(\mathcal{W}_0(\tau_n), \dots, \mathcal{W}_n(\tau_n)) \overset{law}{=} (W_0, \dots, W_n)$ under $\mathbb{P}(\cdot \mid \zeta = n)$

The main difficulty is that this conditioning is "non local". To make it "local" we we are going to use the so-called cycle lemma.

End of lecture 8