Exercise 1 for February 29th

Exercise. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. standard Gaussian $\mathcal{N}(0,1)$ random variables. Set $S_0 = 0$ and for $n \geq 1$ set $S_n = X_1 + \cdots + X_n$. Fix $a > 0$. Show that $S_n - an$ under the conditional law $\mathbb{P}(\cdot \mid S_n > an)$ converges in distribution as $n \to \infty$.

Remark. Equivalently, this amounts to showing that Z_n converges in distribution, where Z_n is a random variable with law characterised by the identity

$$\mathbb{E}[f(Z_n)] = \mathbb{E}[f(S_n - an) \mid S_n > an]$$

for every $f \geq 0$ measurable.

<u>Solution</u>: We use cdf's. For $x \geq 0$:

$$\mathbb{P}(S_n - an \geq x \mid S_n > an) = \frac{\mathbb{P}(S_n \geq an + x)}{\mathbb{P}(S_n > an)}$$

Since $\frac{S_n}{\sqrt{n}} \overset{(d)}{=} N(0,1)$, we have $\mathbb{P}(S_n \geq an + x) = \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{n} + \frac{x}{\sqrt{n}}}^{\infty} e^{-y^2/2}\, dy$

By the change of variables $y = a\sqrt{n} + \frac{z}{\sqrt{n}}$, we get

$$\mathbb{P}(S_n \geq an + x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-(a\sqrt{n} + \frac{z}{\sqrt{n}})^2/2} \frac{dz}{\sqrt{n}}$$

$$= \frac{1}{\sqrt{2\pi n}} e^{-a^2 n} \int_x^{\infty} e^{-az - z^2/2n}\, dz$$

But $\int_x^{\infty} e^{-az - z^2/2n}\, dz \xrightarrow[n\to\infty]{} \int_x^{\infty} e^{-az}\, dz$ by Dominated Convergence.

Thus $\mathbb{P}(S_n \geq an + x \mid S_n > an) \xrightarrow[n\to\infty]{} \dfrac{\int_x^{\infty} e^{-az}\, dz}{\int_0^{\infty} e^{-az}\, dz} = e^{-ax}$

Thus convergence in distribution holds towards an $\mathrm{Exp}(a)$ r.v.

Remarks.
- Performing a change of variables so that the domain of integration does not depend on n is a useful idea
- Alternatively, one could have used $\int_z^{\infty} e^{-u^2/2}\, du \underset{z\to\infty}{\sim} \frac{1}{z} e^{-z^2/2}$