Exercise 1 for February 29th

Exercise. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. standard Gaussian $\mathcal{N}(0,1)$ random variables. Set $S_0 = 0$ and for $n \geq 1$ set $S_n = X_1 + \cdots + X_n$. Fix $a > 0$. Show that $S_n - an$ under the conditional law $\mathbb{P}(\,\cdot \mid S_n > an)$ converges in distribution as $n \to \infty$.

Remark. Equivalently, this amounts to showing that Z_n converges in distribution, where Z_n is a random variable with law characterised by the identity

$$\mathbb{E}\left[f(Z_n)\right] = \mathbb{E}\left[f(S_n - an) \mid S_n > an\right]$$

for every $f \geq 0$ measurable.