Exercise 2 for March 7th

Exercise. Let X be a real-valued random variable. What can be said on the asymptotic behavior of $\mathbb{P}(X \geq x)$ as $x \to \infty$ in the following two cases?

(1) X satisfies $\mathbb{E}\left[e^{tX}\right] < \infty$ for every $t \in \mathbb{R}$ (Cramer condition)

(2) X satisfies $\mathbb{E}\left[e^{tX}\right] < \infty$ for t in a neighborhood around 0 ("local" Cramer condition).

Remark. The question is formulated in an open way, there is no unique answer.

(1) For every $t>0$, $\mathbb{P}(X \geq x) \leq e^{-tx}\, \mathbb{E}[e^{tX}]$

Thus $\limsup_{x \to \infty} \dfrac{\ln \mathbb{P}(X \geq x)}{x} \leq -t$, and this for every $t>0$.

thus the limsup is $-\infty$.

Equivalently, $\mathbb{P}(X \geq x) = \exp(-\omega(x))$

where $f(x) = \omega(g(x))$ means $\dfrac{f(x)}{g(x)} \to \infty$.

Conversely, recall that for $F \geq 0$, increasing:

$$\mathbb{E}[F(X)\mathbb{1}_{X\geq 0}] = \mathbb{E}\Big[\mathbb{1}_{X\geq 0}\int_0^X F'(s)\,ds\Big] + F(0)\,\mathbb{P}(X\geq 0) = \mathbb{E}\Big[\int_0^\infty F'(s)\,\mathbb{1}_{0\leq s\leq X}\Big] + F(0)\,\mathbb{P}(X\geq 0)$$

$$= \int_0^\infty F'(s)\,\mathbb{P}(X\geq s)\,ds + F(0)\,\mathbb{P}(X\geq 0)$$

so for $t \geq 0$

$$\mathbb{E}[e^{tX}] = \mathbb{P}(X\geq 0) + \int_0^\infty t\,e^{ts}\,\mathbb{P}(X\geq s)\,ds.$$

If $\mathbb{P}(X \geq s) = \exp(-\omega(s))$ then $\mathbb{E}[e^{tX}] < \infty$

(for $t<0$ we need information on $\mathbb{P}(X \leq -x)$ as $x \to \infty$)

(2) Similarly, we get $\mathbb{P}(X \geq x) = e^{-O(x)}$

(i.e $\exists c > 0$ s.t $\mathbb{P}(X \geq x) \leq e^{-cx}$ for every $x \geq 0$)

Conversely, if $\mathbb{P}(X \geq x) = e^{-O(x)}$ then

$\mathbb{E}[e^{tX}] < \infty$ for $t > 0$ small enough,

$\rightsquigarrow$ Cramér's conditions are about exponential decay of tail probabilities ("light tails").