Exercise 3 for March 14th

Exercise. Let $(X_i)_{i\geq 1}$ be i.i.d. random variables. Assume that X_1 is integer-valued, aperiodic, satisfies Cramer condition, with $\mathbb{E}[X_1]=0$ and has positive variance. Set $S_n = X_1+\cdots+X_n$. Show that

$$\frac{1}{n}\sum_{k=1}^{n}|X_k| \qquad \text{under} \qquad \mathbb{P}(\cdot \mid S_n=0)$$

converges in probability.

Set $c = \mathbb{E}[|X_1|]$.

Write $\mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^{n}|X_k| - c\right| \geq \varepsilon \,\middle|\, S_n=0\right) \leq \frac{1}{\mathbb{P}(S_n=0)} \mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^{n}|X_k| - c\right| \geq \varepsilon\right)$

by the LCLT, $\mathbb{P}(S_n=0) \sim \frac{c}{\sqrt{n}}$

To control the numerator, 2 possibilities.

Solution 1

It is a simple matter to check that $|X_1|$ and $-|X_1|$ satisfy Cramer's assumption, so that $\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}|X_k| \geq c+\varepsilon\right)$ and $\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}|X_k| \leq c-\varepsilon\right)$ decay exponentially fast, which gives the result

Solution 2

By Bienaymé-Tchebychev's inequality, $\mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^{n}|X_k| - c\right| \geq \varepsilon\right) \leq \frac{1}{n\varepsilon^2} \mathrm{Var}(|X_1|)$

so $\mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^{n}|X_k| - c\right| \geq \varepsilon \,\middle|\, S_n=0\right) \leq \frac{c}{\sqrt{n}}$

Remark In solution 2 we only need finite variance on X_1 (not Cramer's condition)