Exercise 4 for April 11

Exercise. Let $(X_i)_{i\geq 1}$ be i.i.d. random variables. Assume that X_1 is integer-valued non-constant, aperiodic, with $\mathbb{E}\left[X_1^2\right] < \infty$ and $\mathbb{E}\left[X_1\right] = 0$ Set $S_n = X_1 + \cdots + X_n$.

(1) Show that $\dfrac{\max(X_1, \ldots, X_n)}{\sqrt{n}}$ converges to 0 in probability.

(2) Show that the same result holds under the conditional probability $\mathbb{P}(\cdot \mid S_n = 0)$.

Hint. First show that $\dfrac{\max(X_1, \ldots, X_{\lfloor n/2 \rfloor})}{\sqrt{n}}$ converges to 0 in probability under the conditional probability $\mathbb{P}(\cdot \mid S_n = 0)$.

(1) Fix $\varepsilon>0$. We have $\mathbb{P}\left(\left|\frac{\max(X_1,\ldots,X_n)}{\sqrt{n}}\right| \geq \varepsilon\right) \leq \mathbb{P}\left(\frac{\max(|X_1|,\ldots,|X_n|)}{\sqrt{n}} \geq \varepsilon\right) = 1 - \mathbb{P}(|X_1| < \varepsilon\sqrt{n})^n = 1 - \exp\left(n\ln(1-\mathbb{P}(|X_1| \geq \varepsilon\sqrt{n}))\right)$

since $\ln(1-x) \underset{x\to 0}{\sim} -x$, it suffices to show that $n\mathbb{P}(|X_1| \geq \varepsilon\sqrt{n}) \xrightarrow[n\to\infty]{} 0$.

To see this, write $n\mathbb{P}(|X_1| \geq \varepsilon\sqrt{n}) = n\mathbb{P}(X_1^2 \geq \varepsilon^2 n) \leq \frac{1}{\varepsilon^2}\mathbb{E}[X_1^2 \mathbb{1}_{X_1^2 \geq \varepsilon^2 n}] \xrightarrow[n\to\infty]{} 0$ by dominated convergence

(2) Since $\mathbb{P}\left(\frac{\max(X_1,\ldots,X_n)}{\sqrt{n}} \geq \varepsilon \,\middle|\, S_n=0\right) \leq \mathbb{P}\left(\frac{\max(X_1,\ldots,X_{\lfloor n/2\rfloor})}{\sqrt{n}} \geq \varepsilon \,\middle|\, S_n=0\right) + \mathbb{P}\left(\frac{\max(X_{\lfloor n/2\rfloor+1},\ldots,X_n)}{\sqrt{n}} \geq \varepsilon \,\middle|\, S_n=0\right)$

and since $(X_{\lfloor n/2\rfloor+1},\ldots,X_n)$ has the same law as $(X_1,\ldots,X_{n-\lfloor n/2\rfloor})$ under $\mathbb{P}(\cdot \mid S_n=0)$, it suffices to show the hint (formally we should do it with $\lfloor n/2\rfloor+1$, but the reasoning is the same)

For this the idea is to use an absolute continuity argument by writing, using the Markov property,

$$\mathbb{P}\left(\frac{\max(X_1,\ldots,X_{\lfloor n/2\rfloor})}{\sqrt{n}} \geq \varepsilon \,\middle|\, S_n=0\right) = \mathbb{E}\left[\mathbb{1}_{\frac{\max(X_1,\ldots,X_{\lfloor n/2\rfloor})}{\sqrt{n}} \geq \varepsilon} \frac{\phi_{n-\lfloor n/2\rfloor}(-S_{\lfloor n/2\rfloor})}{\phi_n(0)}\right]$$

with $\phi_n(k) = \mathbb{P}(S_n = k)$. By the local central limit theorem, $\exists\, C>0$ s.t $\forall n \geq 1$, $\forall k \in \mathbb{Z}$:

- $\phi_{n-\lfloor n/2\rfloor}(k) \leq \dfrac{C}{\sqrt{n}}$
- $\phi_n(0) \geq \dfrac{1}{C\sqrt{n}}$.

Thus $\mathbb{P}\left(\frac{\max(X_1,\ldots,X_{\lfloor n/2\rfloor})}{\sqrt{n}} \geq \varepsilon \,\middle|\, S_n=0\right) \leq C^2\,\mathbb{P}\left(\frac{\max(X_1,\ldots,X_{\lfloor n/2\rfloor})}{\sqrt{n}} \geq \varepsilon\right) \xrightarrow[n\to\infty]{} 0$ by 1).