Exercise 5 for April 18

Exercise. Let X be a real-valued random variable. Assume that for fixed $T > 0, c > 0, \beta > 2$ we have

$$\mathbb{P}(X \in [u, u+T)) \underset{u\to\infty}{\sim} \frac{c}{u^{1+\beta}}.$$

Show that

$$\mathbb{P}(X \geq u) \underset{u\to\infty}{\sim} \frac{c}{\beta T}\frac{1}{u^{\beta}}.$$

Write $\mathbb{P}(X_1 \geq u) = \sum_{n=0}^{\infty} \mathbb{P}(X_1 \in [u+nT, u+(n+1)T)) \underset{u\to\infty}{\sim} \sum_{n=0}^{\infty} \frac{c}{(u+nT)^{1+\beta}}$

Then $\sum_{n=0}^{\infty} \frac{c}{(u+nT)^{\beta}} = \int_0^\infty \frac{c}{(u+\lfloor x \rfloor T)^{1+\beta}} dx$

$$= \int_0^\infty \frac{c}{(u+\lfloor xu \rfloor T)^{1+\beta}} u\,dx = \frac{1}{u^\beta}\int_0^\infty \frac{c}{\left(1+\frac{\lfloor xu \rfloor}{u}T\right)^{1+\beta}} dx.$$

By dominated convergence, we readily get $\int_0^\infty \frac{c}{\left(1+\frac{\lfloor xu \rfloor}{u}T\right)^{1+\beta}} dx \xrightarrow[u\to\infty]{} \int_0^\infty \frac{c}{(1+xT)^{1+\beta}} dx = \frac{c}{\beta T}.$