Exercise 6 for April 25

Exercise. This exercise is made of two independent questions.

(1) In the proof of the Theorem of Lecture 5, can one replace $\ln(m)^3$ by $\ln(m)$ (and by adding some constants when needed)?

(2) Assume that X satisfies assumption (H_Δ) (with $T < \infty$ or $T = \infty$). Let X_1, X_2 be independent with same law as X. Is it true that

$$\mathbb{P}(X_1 + X_2 \geq u) \underset{u\to\infty}{\sim} 2\mathbb{P}(X_1 \geq u)?$$

1) No, because of the step showing that

$$Q_1^{m,n} = \mathbb{P}\Big(S_n \in \Delta_m,\ X_n \geq \bar{m},\ \max_{1\leq k\leq n-1} X_k < \bar{m},\ S_{n-1} > \frac{m}{\ln(m)}\Big) = o\Big(\frac{n}{m^{1+\beta}}\Big)$$

Indeed, by the maximal inequality, $Q_1^{m,n} \leq \mathbb{P}\Big(S_{n-1} > \frac{m}{\ln(m)},\ \max_{1\leq k\leq n-1} X_k < \bar{m}\Big) \leq K\exp\Big(-\frac{m}{\ln(m)\,\bar{m}}\Big)$, so if we take $\bar{m} = \frac{m}{L\ln(m)}$ with $L>0$ being some constant, we get

$$Q_1^{m,n} \leq K\exp(-L)$$ which is not enough to get $o\Big(\frac{n}{m^{1+\beta}}\Big)$.

2) Yes! Since (H_Δ) for $T<\infty$ implies (H_Δ) for $T=\infty$, it is enough to show the result when $\mathbb{P}(X_1\geq u) \overset{(*)}{\sim} \frac{C}{u^\beta}$ for $\beta>2$. To see it write for $u>0$

$$\mathbb{P}(X_1+X_2\geq u) = \mathbb{P}(X_1+X_2\geq u,\ X_1\leq u/2) + \mathbb{P}(X_1+X_2\geq u,\ X_2\leq u/2) + \mathbb{P}(X_1\geq \tfrac{u}{2},\ X_2\geq\tfrac{u}{2})$$
$$= 2\,\mathbb{P}(X_1+X_2\geq u,\ X_1\leq u/2) + \mathbb{P}(X_2\geq \tfrac{u}{2})^2$$

By $(*)$ $\mathbb{P}(X_1\geq\frac{u}{2})^2 = o(\mathbb{P}(X_1\geq u))$, so it is enough to show

$$\mathbb{P}(X_1+X_2\geq u,\ X_1\leq\tfrac{u}{2}) \underset{u\to\infty}{\sim} \mathbb{P}(X_1\geq u)$$

To see this, write $$\frac{\mathbb{P}(X_1+X_2\geq u,\ X_1\leq\frac{u}{2})}{\mathbb{P}(X_1\geq u)} = \int_{-\infty}^{\infty}\frac{\mathbb{P}(X\geq u-x)}{\mathbb{P}(X\geq u)}\mathbb{1}_{x\leq\frac{u}{2}}\,\mathbb{P}_X(dx)$$

where X has law X_1. We apply dominated convergence:

- for $x\in\mathbb{R}$, $\frac{\mathbb{P}(X\geq u-x)}{\mathbb{P}(X\geq u)}\mathbb{1}_{x\leq\frac{u}{2}} \xrightarrow[u\to\infty]{} 1$
- $\frac{\mathbb{P}(X\geq u-x)}{\mathbb{P}(X\geq u)}\mathbb{1}_{x\leq\frac{u}{2}} \leq \frac{\mathbb{P}(X\geq u/2)}{\mathbb{P}(X\geq u)} \leq C$ by $(*)$, which is $\mathbb{P}_X(dx)$ integrable.

This completes the proof